COLUMNS AND STRUTS

W. ALEXANDER

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THE THEORY AND PRACTICAL DESIGN

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COLUMNS AND STRUTS

THEORY AND PRACTICAL DESIGN

WITH EXAMPLES WORKED OUT

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PREFACE

THIS work is the result of investigation by the Author, found to be necessary, during the course of his professional work, in connexion with the practical design of large struts, some of them as much as two hundred feet in length.

Euler's column formula was the only one in existence, based on purely theoretical reasoning. That formula, and the other partially empirical formulæ based on it, owing to some factors being rejected in their construction, lead to the conclusion that a strut of uniform section throughout, passes into a state of indifferent equilibrium when it commences to bend under an axial load, and therefore that no relation can be determined between the load and the amount of deflection. This conclusion, although not true, has been generally accepted, and has formed a barrier to progress in the investigation of column and strut phenomena.

The investigation herein is carried out on lines somewhat different from the usual course. The true relations between load and deflection are demonstrated, and the resulting equations are based on the maximum permissible amount of deflection, that limiting amount being, in turn, determined from the maximum permissible fibre stress intensity.

Following the same course, the nature of the shearing or web stresses is shown, and their amounts determined directly from the equations to the curves of deflection, and not, as has so far been the practice, from an imaginary equivalent transverse load.

It is shown that the form of the cross section has a great effect on the strength of a strut, and the efficiency of common sections is discussed in detail. The practical equations derived from the investigation allow for variations in the form of section, and all the usual corresponding empirical constants or factors are avoided.

In the discussion of this subject, it has been necessary to enter into arguments and use analytical processes of a somewhat abstruse nature, but these are stated as clearly as possible, so that the methods of arriving at the final equations may be understood, without the necessity for going through all the steps in detail.

All the column formulæ in existence deal only with cases in which the section is uniform from end to end. The result of the most common forms of tapering in a column is investigated herein, and the general effect of tapering is indicated.

A number of practical matters affecting details of design, end connexions, diaphragms, etc., are discussed.

As a valuable object lesson, a somewhat detailed investigation has been entered into of the struts which were the first to fail in the Quebec bridge, which collapsed in 1907.

In the final chapter a summary is given of the Notation used and the Principal Equations required for practical designing, and a number of problems in design are worked out to illustrate the correct application of the equations.

COLUMNS AND STRUTS

CHAPTER I.

INTRODUCTORY.

ABOUT one-half of the members of which structures, such as bridges, roofs, towers, stages, etc., are formed have to resist compressive stresses, and act as struts. In exceptional cases, such as suspension bridges, the proportion may be less than half, but these cases are very much in a minority.

Although, owing to their nature, struts are the most important members, the subject of their design is the least understood. An examination of existing structures will show startling differences in the quantities of material used in the construction of struts, even under similar conditions. If the heavier of such struts have not unnecessarily great margins of safety for carrying their loads, the lighter ones must have margins much too small. On the other hand, if the lighter struts have a sufficient margin, there must be great waste of material in the heavier ones. It does not always follow that a heavier strut is stronger, and in fact, the addition of material, unless properly placed and distributed, may actually weaken the strut as a whole. Even if the strut was not weakened, the extra material might be useless, and not only involve extra and unnecessary cost in the strut itself, but, on account of the extra weight to be carried, the stresses in the other members of the structure of which it formed part would be increased, and consequently involve further weight and cost, or, if these other members were not increased, the net carrying capacity of the whole structure would be reduced. In a very large structure unnecessary weight in the struts, to the extent of only two or three per cent., would involve corresponding cost of a very considerable amount, and the extra and useless weight might amount to a very large proportion of the load which the structure was designed to carry.

Many formulæ exist and are used for determining the strength of struts, or their carrying capacity, but are not generally understood. These formulæ give widely differing results, and, in general practice, it is by no means an unknown thing for a designer to adopt a certain formula, because "it gives a more favourable result," meaning that the strut designed thereby will be lighter than if other formulæ were used, and consequently it will be cheaper. If properly understood, many of the so-called formulæ would never be used.

For struts of what may be termed moderate dimensions, refinements in theoretical calculations are unnecessary, as such struts are generally formed of rolled sections, and as it is only possible to obtain these in certain market sizes, and as, in the case of most of them, tests have been made to ascertain their carrying capacity, it is easy to select a suitable one. In built-up sections, however, the theoretical section can be more closely approximated to by the selection of suitable bars which, when combined, form the complete section, and the larger the strut the more closely can the theoretical requirements be carried out, and the more necessary does it become to make certain that the theories or formulæ made use of are true and based on correct data. This is the case when dealing with large structures, both on account of the more serious consequences of a failure, and because modern conditions of commercial competition render it more, and increasingly, necessary to avoid any waste of material; and, while in a small strut it might cost very much more to form a section closely to that indicated by theory than would be saved in material by doing so, and therefore the reverse of economical to attempt it, the waste of a small percentage of material in a large structure might involve cost which would make all the difference in the commercial result of an undertaking.

The necessity for correct design is emphasized by the fact that there are on record cases where structures, after carrying loads for years, have collapsed, through the failure of struts, owing merely to slight changes in the distribution of the loads without their totals being appreciably increased. This shows that structures have been designed and constructed, containing struts which were dangerously weak, and it is probable that there are many others in existence.

The importance of the subject becomes greater every year. So far as masonry columns are concerned, their proportions are never very slender, and the question of the quantity of material used is seldom of great importance. Owing to the somewhat limited dimensions of timber readily obtainable, and the difficulty of making effective joints between the various members, and also to the fact of the material being of a perishable nature, there has been no great increase in the magnitude of timber structures. With the introduction of wrought iron as a constructive material, conditions became much modified, and the later introduction of steel has rendered possible the erection of the huge modern structures, containing struts, in some instances hundreds of feet in length, and of enormous weight.

During the last few years the subject has been brought to notice with great force owing to the collapse of the great steel bridge, of 1800 feet span, which was being constructed over the river St. Lawrence at Quebec. That bridge fell, before being completed, owing to the failure of some of the members of the main compression booms of the cantilevers, although the direct stresses on these members were, at the time, little more than two-thirds of what they would have been if the bridge had been completed and carrying its load.

In the published report, of 1908, of the Royal Commission on the Quebec Bridge Failure, on page 10, there is the following statement :—" The professional knowledge of the present day, concerning the action of steel columns under load, is not sufficient to enable engineers to economically design such structures as the Quebec Bridge. A bridge that will unquestionably be safe can be built, but in the present state of professional knowledge a considerably larger amount of material would have to be used than might be required if our knowledge were more exact."

That statement appears to be somewhat illogical, as, if it is not possible to determine the correct amount of metal required in a strut, how could it be known that the amount used was greater than necessary? On the other hand, if it could be known that the amount used was greater than necessary, why put in so much?

The author does not agree with the statement in its entirety as relating to the particular struts which failed, as the reason for their failure can be shown, and their design is considered in some detail in Chapter XII. At the same time, if the statement is, to any extent, generally correct, the necessity is very great for further research, and also for the elucidation of some existing theories.

The strength of large struts cannot, for practical reasons, be determined by experiment. The conditions under which they are used vary so much that great varieties of forms of cross-section are necessitated, and the results of experiments on one form of section do not necessarily supply data suitable for designing other forms; in fact, experiments show that they do not, and as it would be necessary to make tests of every new form of section devised, this method of research becomes prohibitive on account of cost.

As the result of experiments, it is comparatively easy to be sure, within narrow limits, what load a column of a certain cross-section and length will carry, even if not exactly the same as one of those tested, so long as it falls within the limits or range covered by the experiments, and by comparison it is possible to construct columns with safety within that range; but if it was attempted to extend operations outside of the range covered by the experimental data, the result might by chance be near the mark, but just as likely very far from it.

Even if the results of experiments appear to indicate that certain derived formulæ give close approximations, such formulæ cannot always be depended upon to give equally close results beyond that range of observations, and as experiments on struts must be confined to samples which are small compared with those used in large structures, it is quite necessary that what theories are used should be true and correctly applied.

In designing important structures or members such as struts, no engineer should make use of formulæ, especially for unusual cases, unless he understands their meaning and generally their derivation. The mathematical processes involved in investigating the curves assumed by a strut bent under a load are of a highly transcendental nature, and the engineer or designer who ordinarily makes use of the derived formulæ very often has not the time to go through all the steps in detail.

An attempt has been made in the following chapters to describe these curves and express their equations in such a way that a reader, if he has a general idea of what a differential or an integral is, may obtain a direct knowledge of their nature and properties and understand in what manner the working formulæ are derived, also realizing to some extent the limitations and defects of the most important of the empirical formulæ in use.

In the designing of struts, probably more than concerning any other form of member, it is necessary to combine practical and theoretical knowledge. Practical knowledge is necessary to determine the forms which lend themselves best to economical construction and maintenance, and also to realize the conditions under which the struts will have to perform their duties, as well as to obtain the necessary data. Theoretical knowledge is necessary, to make use of the data and to determine stresses, to arrange the proper distribution of the material, design the end connexions to ensure a correct distribution of the load, etc. It is possible to design many parts of structures with the aid of very little theoretical knowledge, but a strut is one thing which cannot possibly be designed without the understanding of formulæ, the production of which entails very considerable technical processes.

Much discussion takes place at times as to whether certain formulæ are empirical or absolute. According to some arguments, any formula is empirical if it makes use of data obtained by experiment, but this shows a want of appreciation of what theory is. It is impossible to apply theory in any way without making use of data, and data can only be obtained by experiment. Data are useless without theory, while by making use of theory, results outside of the range of existing experience can be predicted with certainty. The ultimate stress intensity of steel, say 30 tons per square inch, could only be determined by experiment, but no one could therefore say that to determine the ultimate strength of a group of five bars of such steel, each having one square inch of sectional area, the process of multiplying five by thirty, giving one hundred and fifty tons, was an empirical one.

It may be taken as absolute that, if the principles on which any formula is deduced are correct, and if when properly applied the predicted results are not obtained in practice, the fault is not in the theory but in the data on which the deduced formula is based. It may be that the data are incorrect or insufficient.

Sometimes it is practically impossible to prepare formulæ simple enough to be used in ordinary designing. Then, the the results of experiments may be considered, and, by trial and error or other means, some kind of an algebraic expression constructed which, used as a formula, gives approximately correct results, agreeing with the results of the experiments within the practical range and limits of dimensions, proportions of parts, conditions of loading, etc., covered by the experiments. Such formulæ are purely empirical, and there is no guarantee that they will give anything like correct results except within the observed range.

Again, in developing formulæ, it is sometimes found that the processes of analysis are so intricate or obscure that it may so happen that the construction of an absolute formula requires the use of equations, or the summation of series, etc., involving functions of a nature too intricate for general use ; but by substituting other functions of a simpler nature, the resulting formula contains much simpler terms and yet gives results closely approximating to the true ones, sometimes over a very wide range, and the amount of error at any part of the range can be determined. Such a formula is only partially empirical.

It is always safe to use an absolute formula, even far beyond the range previously common. It is never really safe to use a purely empirical formula outside the range of the experiments from which it was derived. In using a partially empirical formula, it is always advisable to know its limits, and if properly used it may be quite as good and useful as the more intricate absolute formula for which it has been substituted; and, in fact, if critically examined, the majority of engineering formulæ will be found to be partially empirical.

The subject of the correct design of struts is not an easy one, and no very simple formulæ can be constructed to cover all cases, and at the same time secure economy.

The equations given herein are comparatively simple and not difficult to use. Their use should take up no more time in calculation than the formulæ generally used, as in all cases it is necessary to proceed tentatively and arrive at final results by processes of trial and error.

In carrying out the construction of a very large bridge or other work the employment of a few extra days, or even weeks, in properly calculating the strength of the struts will involve a cost very trifling in comparison with the economy secured and the satisfaction felt in the results.

CHAPTER II.

ANALYSIS OF VARIOUS EXISTING FORMULÆ.

IN calculating the strength of a tension member, it is usual to consider simply the product of the number of units of sectional area and the allowable stress intensity, which is considered as the strength. If, as is generally the case, the load is applied symmetrically about the axis of the member, the resulting stresses are distributed uniformly over the section. If, however, the mean or centre line of the application of the load or external force does not coincide with the axis of the member, bending moments are produced in addition to the direct stress. These bending moments cause a non-uniform distribution of stress, and as a result the member will bend or deflect. It is easily seen that this deflection will take place in such a direction that the axis of the member moves to a position nearer to the line of application of the load, and the amount of the bending moment is reduced. As the member bends, it exerts a resistance to the bending action, increasing with the amount of deflection and, as this resistance increases while the bending moment decreases, the two will eventually become equal, and the forces will then be in equilibrium. This state of equilibrium will occur before the deflection reaches the amount by which the line of application of the load originally departed from the axis. A tension member will always deflect in such a direction that the resulting stresses tend to become more uniformly distributed over the section.

In a strut, or compression member, also, if the axis is perfectly straight, and the line of application of the external force coincides with the axis, the resulting stresses are uniformly distributed, while if the line of the external force departs from the axis, there will be bending moments causing the strut to deflect, but in this case the deflection will commence in such a direction that the axis moves further from the line of the external force, causing an increase in the bending moment, and consequently a further increase in the bending, and so on. This bending will continue to increase until, if the strut is strong enough, the resistance which it offers to the bending action equals the amount of the bending moment, or, if not strong enough, until it collapses.

Should the axis be straight and the direct external force coincide with it, but the strut afterwards be caused to deflect by being pushed out of line by some supplementary force, the axis will no longer coincide with the line of the direct force, which will then produce a bending moment. If the supplementary force is now removed, the strut exerts a tendency to recover its original straight form, and it depends upon its strength whether it will recover, or whether the direct force is great enough to hold it in its bent form, or to increase the bending.

Contrasted with a tension member, a strut will always bend, under a direct load, in such a direction that the resulting stresses tend to become less evenly distributed over its section. In practice this fact is of the greatest importance, and must be taken into account, and it is the chief factor to be considered, not always because the stresses due to the bending moments, within practical limits, are great, but because they always increase the stresses due to the direct action of the load, and after bending commences, the stresses due thereto increase at a very rapid rate.

As all formulæ relating to the strength of struts are based upon certain assigned limits of curvature and therefore of deflection under load, we will now consider the usual formulæ relating to the deflection of a bar under the action of external forces.

In Fig. 1, let A B be a bar fixed at A, so that its axis X X always remains horizontal at that point. If A B is straight and lies horizontally when free from load, a vertical load P applied at B will bend it into a curved form such as A B_1 It will be seen that the horizontal distance S_1 from A to B_1 is less than A B or S, and therefore the bending moment at A is P × S_1 which is less than P × S. If absolute equations are required to give the real deflection v_1 great complexity is involved. As, however, from a practical point of view, the amounts of deflection allowable or obtained in such beams are small compared with their lengths, and the difference between S and S_1 is very much smaller than the amount of deflection, no sensible error is introduced if S be used instead of S_1, and the bending moment called P × S. Any error caused by using the substituted value P × S in calculating the deflection v is much less than the probable variation of v owing to non-homogeneity of material, variation in elasticity, etc.

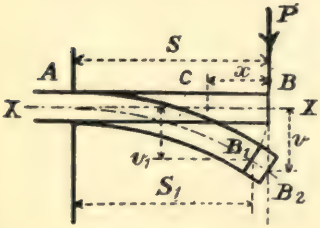


FIG. 1.

By substituting the value of S for S_1, the equations become very much simplified, and although they are really empirical they are quite as suitable for predicting the deflection, etc., as if the absolute equations were employed.

Using S for S_1 is the same thing as assuming that the distance A B measured along the curve, and which is equal to S, is also equal to S_1, or that the length of an arc is equal to its chord.

If S_1 did remain equal to S as the bar deflected, the point B would not move to B_1, but to some other point B_2, and as A B_2 must be longer than A B, it follows that as the bar deflected it would have to stretch out or increase in length. In reality it does not do so, but the assumption causes no difficulty if it is merely required to predict v, because, working down from comparatively large quantities such as S, to arrive at small results such as v, the error becomes negligible in proportion to the actual values of v, which occur when beams or cantilevers deflect. The proportion of deflection v, compared with S, as indicated in Fig. 1, is much exaggerated for the sake of clearness. If, however, it was attempted to calculate the deflection of a steel spring, bent to the extent indicated, or more so, the resulting errors would not be negligible, but would be very great.

The formula generally used, and derived on the above assumptions, for the value of v, Fig. 1, is

$$v = \frac{f S^2}{3 E \delta} \quad . \quad . \quad . \qquad (1)$$

where f is the maximum fibre stress intensity, E the modulus of elasticity, and δ the distance of the extreme fibre from the neutral axis of the section.

As showing that this formula is only approximately true within small limits, it may be pointed out that it represents v as varying directly as f, and therefore (as f varies directly as P) v should also vary directly as P, and if P was continually increased, v should also continually increase, and it would eventually become even greater than the length S of the cantilever itself, which is, of course, impossible.

The formula also represents that, for constant values of E and δ, that is, for constant section and material and constant f, the deflection v should vary as S^2. This, again, is not strictly true, and although it is of no consequence in considering the deflection of beams, the fact must be borne in mind, as it affects the accuracy of certain conclusions which are generally drawn from similar expressions occurring in existing column formulæ, some of which are noticed later.

If the preceding problems were reversed, and an attempt made, by transposing the empirical formula (1), to calculate the length of a bar which would deflect to a given extent, v, it does not follow that the error in the result would still be negligible. It would be of an appreciable amount, and, as v increased, the error would increase at a rapid rate.

Such a process is the foundation of the majority of column formulæ, and it is necessary to exercise great care in their application. Many writers make use of equations based on such assumptions to determine the character of the deflection curves and their properties. In doing so, the fact that the original equations are not exact is frequently lost sight of, and statements are made, such as that the result of certain manipulations of the formulæ and factors *proves* something, which is then put forward as being a fact. In this way some very erroneous and quite absurd conclusions have been arrived at.

The earliest formula with respect to the relations between the dimensions, loads, etc., of struts, is due to Euler, and it is the only one which was evolved on a purely theoretical basis. It has been used as the foundation of many proposed modified formulæ, to allow for certain variations observed in the results of tests, etc. Other formulæ, of a more empirical nature, were introduced by Tredgold and Hodgkinson, and subsequently modified by Gordon, Rankine, etc.

With regard to a spring, bent by forces as indicated in Fig. 2, Rankine states (Applied Mechanics, art. 319), among other things, that :—

" . . . the radius of curvature is proportional to the perpendicular distance from the line of action of the forces,"

and then gives the equation

$$\rho = \frac{E\,I_0}{M} = \frac{E\,I_0}{x\,P} \quad (3) \quad . \quad (2)$$

afterwards continuing :—

"There is one important proposition which it is necessary to *prove*, and that is the following :—

"THEOREM.—That a spring of a given length and section, to the ends of whose neutral surface a pair of forces are applied, will not be bent if these forces are less than a certain finite magnitude. Let A and B (as in Fig. 2) be the two ends of the spring, to which two opposite and equal forces of the magnitude P are applied, directed towards each other; the spring forming a single arc of the length l, x being the ordinate of any point c, let y be the distance of that ordinate from A.

"The smaller the force P, the more nearly will the arc A c B approach to the straight line A B; and in order to find the smallest value of P which is compatible with any bending of the spring, that force must be computed *on the assumption* that the ordinate at any point is insensibly small compared with the length of the spring, and consequently, *that the length of the arc* A c *does not sensibly differ from that of its abscissa* y. This being the case, the curvature at any point c is to be taken as *sensibly* given by the following equation:—

$$\frac{1}{\rho} = -\frac{d^2 x}{d y^2}$$

which value, being inserted in equation (3), gives

$$-\frac{d^2 x}{d y^2} = \frac{\mathrm{P}}{\mathrm{E\,I_0}} x \qquad (4) \quad . \quad (3)$$

"The integral of this equation is

$$\left.\begin{aligned} x &= a \cdot \sin \frac{y}{c} \\ \text{where} \quad c &= \sqrt{\frac{\mathrm{E\,I_0}}{\mathrm{P}}} \end{aligned}\right\} \qquad (5) \quad . \quad (4)$$

"In order that x may be $= 0$ at the points A and B, it is necessary that when $y = l$, $\frac{y}{c}$ should be $= n\pi$, n being any whole number; and consequently that

$$c = \frac{l}{n\pi} \qquad (6) \quad . \quad . \quad (5)$$

"Now of all the possible values of n, that which gives the least value of P is $n = 1$; whence we find

$$c = \sqrt{\frac{E\,I_0}{P}} = \frac{l}{\pi};\ \text{and } P = \frac{\pi^2 E\,I_0}{l^2}\ (7) \quad (6)$$

and *this finite quantity is the smallest force which will bend the spring* in the manner proposed.—Q. E. D.

"This investigation *proves* the Theorem in question and gives the least bending force P."

The words printed in italics in the foregoing do not appear so in Rankine's treatise, but have been printed so here to call attention to the method of reasoning adopted.

The equation (7) . . (6), above, is that known as Euler's formula, and from first impressions, it might be thought that, owing to the analysis being based on the assumptions to which attention is called, the resulting equation is not exact. Lengths measured along the chord are used instead of along the arc, but as the deflection or versed sine of the arc diminishes, the chord and the arc become more nearly equal, that is, they approach to the same limit; and as the equation only professes to give the limiting value of P when there is no bending or deflection, *it is absolutely correct and in no way approximate.* This is not self-evident, or very clear on a simple inspection of the process, but a proof of the exactness will be found on referring to equation (99), which is a true equation to the curve of the bent spring or strut, and no assumptions have been used in constructing it. The limiting value, when there is no deflection, is given by equation (111),

$$\text{where } L = 2\,S \quad \text{and therefore } L^2 = 4\,S^2 \quad (7)$$

This is exactly the same result as given by Euler's formula, Rankine's interpretation of which is perfectly accurate, namely, that it gives the smallest force which will bend the spring.

Euler's formula applies only to a spring, or strut of uniform section, perfectly homogeneous material, axis perfectly straight, and with the external forces applied directly along the axis. Such a strut is known as an "ideal" strut or column.

The real meaning of Euler's formula is illustrated in Chapter VII., and its connexion with the absolute formulæ for the curves of bent struts is shown.

Much discussion has taken place regarding this formula, but its true meaning appears to have been realized by few. It is generally taken to be a formula giving a value P, which is the crippling or breaking load of a strut, and is very frequently used in that way, whereas it merely gives the load at which deflection would commence. If, by it, an attempt is made to find the breaking load of a column, having a ratio $\frac{L}{r}$ of a small value, or, what is the same thing, of a column having a section large in proportion to its length, a result will be obtained much greater than the resistance of the material to direct compression; and it is then argued that the formula fails for ratios of $\frac{L}{r}$ less than a certain value. If the formula was properly interpreted, it would be realized that the value P obtained was not the theoretical breaking load, but that, for these small values of $\frac{L}{r}$ the strut would not bend under a less load, and would therefore crush before bending commenced.

Although the premises and final equations of Euler's investigation are absolute, the intermediate equations are not so. One of these equations, No. (4), which gives $x = a \cdot \sin \frac{y}{c}$ has been used in one form or another by many writers, and as it is the equation to a curve of sines, they state, without reserve, that the curve of a bent strut is a curve of sines. Even in the report of the Royal Commission on the failure of the Quebec Bridge it is authoritatively stated that it is so. This is an important case of an empirical formula being used to *prove* something which is afterwards put forward as a fact.

For the design of small struts, a sine curve is just as good as the true curve, but so also would be an arc of almost any other curve. Circular arcs and parabolic arcs have been used, and results quite as good obtained. There is no special merit, for the purpose, in a sine curve. The true curve is not a curve of sines, but is the elastic curve sometimes called the Lintearia, or Elastica. The sine curve has none of the properties of the Elastica, and the only thing that can be said for the sine curve is, that when the versed sine diminishes and finally disappears, the factor in the equation, denoting the length, approaches the same limit as the corresponding factor does in the equations to the Elastica. Many other curves might be constructed which would have this property. The curvature at any point in the sine curve does not vary as its distance from the chord, and therefore a spring bent into a sine curve would not be in equilibrium under two external forces only, acting along the chord.

In a case of this kind it is unadvisable to make emphatic statements such as that a curve is of a certain form when it is only an approximation. The statement may lead to no trouble in practice, but it is likely to give students false ideas, as in the majority of cases, when put into print, such statements are accepted by readers, unless they go to the trouble of examining all the data and steps in the reasoning.

It is sometimes stated that Euler's formula not only fails for struts having a small ratio $\frac{L}{r}$ or what are termed short struts, but also that for very long columns it gives results too small; and experiments show that such columns often carry loads greater than the formula would indicate. This, again, is only what might be expected, because the value of P obtained is the load under which the strut should commence to bend, and as very long struts will bend to an appreciable extent before collapsing, their ultimate load, under ideal conditions, is greater than that value of P, under which bending commences, and also because an average value of E is used, which must be exceeded in some struts.

In close connexion with the foregoing, another misapprehension is very general, namely, that the load P which will commence to bend a strut will be sufficient to bend it to any extent, or in other words, that as the strut is bent more and more its resistance to the bending action does not increase, but remains constant however much it may be bent. This is another result of considering Euler's formula as giving the crippling load, and the curve being a curve of sines. If used in this way and the formula is transposed to give the relation between P and the deflection v, it would appear that P was constant for any value of v. This is a fallacy, and it will be quite apparent that it is so, because when the spring, Fig. 31, begins to bend, v increases and continues to do so until the form Fig. 33 is reached, after which v actually decreases. Now the spring cannot behave according to one law at one time and afterwards according to another law, and P is not constant. The formula, thus used to determine v, indicates that v may increase to any extent, whereas it really has a limiting maximum. This false conclusion is due to the same assumption that the length of the arc is equal to the length of the chord, and it requires that, as the strut bends, its ends remain at a constant distance apart; therefore the length of the strut would have to increase as the deflection increased.

If a strut deflected in the manner thus assumed, a diagram showing the ratio of deflection to load would be as in Fig. 42. Up to the point where the deflection commenced the deflection curve would be a straight line, Og, coinciding with the abscissa, and it would then rise vertically along gy to an indefinite height. The true deflection curve is indicated in the same Fig., and described in Chapter VII.

It will be seen that at g the true curve is tangential to the line gy, and for small deflections may be said to coincide with it, and therefore the formula be quite accurate enough for practical purposes. This is so within moderate limits. It is not intended to infer that these formulæ are useless, but merely to show their true meaning and point out some false deductions.

It has, further, been stated that Euler's formula is not rational because it only takes bending into account and not the stresses due to the direct load. This is a mistaken argument. Seeing that if the load on the column is kept within the value of P, given by the formula, and P is the smallest load which will cause bending, there can be no bending and therefore no stresses except those due to the direct action of the load. The only way in which the formula can be said to take bending into account is that its use is supposed to ensure the absence of bending.

This formula also makes it appear that the strength of a column of given section varies inversely as the square of the length, but this, although nearly true within certain limits, is not absolutely true, the error being again due to the false assumptions.

Hodgkinson's formulæ for cast-iron columns were deduced from the results of experiments made by himself. He divided these columns into two classes :—

I. Those in which the length is thirty times the diameter, or greater, and his formulæ read :—

For solid cylindrical columns

$$P = A\frac{h^{3\cdot6}}{L^{1\cdot7}} \quad . \quad . \quad . \quad (8)$$

L being the length *in feet*, and h the diameter *in inches* ; and for hollow cylindrical columns

$$P = A\frac{h_1^{3\cdot6} - h_0^{3\cdot6}}{L^{1\cdot7}} \quad . \quad . \quad . \quad (9)$$

L being the length *in feet*, h_1 the external diameter, and h_0 the internal diameter, *in inches*.

In these formulæ A is a constant, and the values determined were:—

						A = Tons
(1)	For solid	columns,	with rounded	ends	.	14·9
(2)	„	„	flat	„	.	44·16
(3)	For hollow	„	rounded	„	.	13·0
(4)	„	„	flat	„	.	44·3

II. Those in which the length is less than thirty times the diameter, when the formula is given as

$$P = \frac{b\,c}{b + \frac{3\,c}{4}} \quad . \quad . \quad . \quad (10)$$

b being the value of P if calculated according to (8) or (9), and $c = 49 \times$ sectional area *in square inches.*

These formulæ are purely empirical, and their use is extremely limited. It would be necessary to make experiments for every new section devised if they were generally used for designing purposes.

Another type of formula, constructed on a theoretical basis, but with empirical constants, was originally developed by Tredgold, and is as follows:—

The length L is supposed to be comparatively great so that bending may occur. Then, referring to Fig. 2, on the section $a\,b$ at the centre of the column, there will be stresses consisting of the direct force P distributed uniformly over the area A of the section, combined with other stresses due to the bending moment P v and not uniformly distributed.

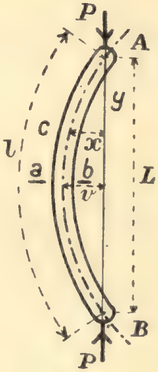


FIG. 2.

Calling p' the intensity of the direct stress

$$p' = \frac{P}{A} \quad . \quad . \quad . \quad (11)$$

Calling p'' the maximum stress intensity on the extreme fibre, due to the bending moment, then

$$p'' \text{ varies as } \frac{P\,v}{B\,h^2}$$

h being the least diameter of the column and it being assumed that the bending will take place in the direction of h, B being the diameter at right angles to h, both measured in the same units as L.

It is then assumed that the deflection varies directly as L^2 and inversely as h, whence

$$v \text{ varies as } \frac{L^2}{h}$$

and as $B\,h^2$ varies as $A\,h$

$$p'' \text{ varies as } \frac{P}{A} \cdot \frac{L^2}{h^2} \qquad \text{as } p' \frac{L^2}{h^2}$$

If f represents the maximum permissible fibre stress intensity, then

$$f = p' + p'' = \frac{P}{A}\left(1 + 4\,a\frac{L^2}{h^2}\right) \qquad . \quad (12)$$

a being a constant, to be determined by experiments.

This equation being transposed gives

$$P = \frac{f\,A}{1 + 4\,a\,\dfrac{L^2}{h^2}} \qquad . \quad . \quad (13)$$

The reason for the constant being given as $4a$ in this equation is that, according to the argument, a strut with ends free is twice as flexible as one having its ends fixed in direction, and therefore the constant relating to the former will be four times as great as for the latter type. Therefore, as the fixed end is the stiffest possible type, the constant therefor will be the least, and calling that constant a the corresponding value for the free-end type will be $4a$.

From the results of Hodgkinson's experiments, Gordon computed the following values of a :—

	f = lbs. per sq. in.	a
Wrought Iron, solid rectangular section	36,000	$\frac{1}{3000}$
Cast iron, hollow cylinder	80,000	$\frac{1}{800}$
„ solid „	80,000	$\frac{1}{400}$

As a strut with rounded or free ends is as flexible as one of the same section, fixed at both ends, and of double the length, and the strength being assumed to vary as the square of the length, the formula for the fixed-end type becomes

$$P = \frac{fA}{1 + a\frac{L^2}{h^2}} \quad . \quad . \quad . \quad (14)$$

This formula is generally known as Gordon's Formula.

Mr. Hodgkinson found that the relative strengths of columns did not vary in accordance with the assumptions, and that the results were affected by the actual dimensions.

It will be seen that the constant in Gordon's formula varies much with the form of the section, for example, being for the solid cylinder double the amount determined for the hollow cylinder. For this reason, Rankine substituted for the factor h^2 the value $12\,r^2$, r being the radius of gyration of the section in the direction of the bending, and as the sections on which the experiments were made were rectangular, $h^2 = 12\,r^2$.

Gordon's formula, as modified by Rankine, then becomes, for free-ended struts

$$\text{for iron,} \qquad P = \frac{fA}{1 + \frac{1}{9000}\frac{L^2}{r^2}} \quad . \quad . \quad (15)$$

$$\text{for mild steel,} \qquad P = \frac{fA}{1 + \frac{1}{7000}\frac{L^2}{r^2}} \quad . \quad . \quad (16)$$

This is generally known as the Rankine, or Rankine-Gordon formula. It will be seen that, here also, it is assumed that the strength varies inversely as the square of the length, and that f varies directly as P, but neither of these assumptions is true, although, for practical purposes, approximately so. It is also assumed that bending will take place in that direction in which r is least for the section. This may by chance be so, but it is more likely to be otherwise.

The Rankine-Gordon formula has probably been used more generally than any other, although the constants employed have been given different values by various authorities. It depends entirely upon these constants, which have to provide for all possible causes of departure from ideal conditions which are not met with in practice, and also for the effects of varying forms of section, etc. All provision for the whole of these is included in the one constant, and the formula, as used, becomes simply the result of attempts to make it agree with the average results of as large a number of experiments as possible, without discrimination.

As in the case of Euler's formula, many variations have been proposed in this formula, to cover observed deviations from the predicted results. For example, Fidler (Practical Treatise on Bridge Construction), has introduced factors providing for eccentricities of loading due to varying elasticity, etc., and suggested modified formulæ. Fidler's work should be read by everyone interested in the design of struts. In addition to a concise description of the usual theories, etc., he has compiled useful tables for facilitating the ready selection of suitable sizes of sections of frequently occurring form.

For determining the stresses in, and curve of, a bent strut, Fidler has also made use of a process, used in calculating deflection of beams, consisting of a method of finding the slope, or tangent, to the curve, from the area of bending moment diagram, and this leads to the conclusion that the curve is a curve of sines. It must be remembered, however, that in evolving the said process, it was assumed that lengths measured along the arc were sensibly equal to those along the chord. Any process in which this assumption is made, at any stage in the reasoning, will lead to the same conclusion.

In reading any work on struts it is necessary, for a thorough understanding of the subject, to remember that none of the formulæ which have, so far, been constructed are exact, and although, as stated before, numerical results may be quite near enough for practical purposes, many of the oft-repeated conclusions, and the calculations given to prove them, are not absolute. Many statements, also, are not sufficiently guarded in their language, and certain processes are frequently described as being *proofs*, *exact determinations*, *mathematical proofs*, etc. ; the fact being apparently overlooked, for the time being, that the original data were only approximate.

Such statements as, that the deflection curve is a curve of sines ; that when a strut commences to bend it passes into a state of unstable equilibrium ; that the deflection will increase indefinitely ; that the flexibility varies as the square of the length ; that the load which will just commence to bend a strut or spring will, without being increased, hold it bent to any extent ; that f or v varies as P, etc., are all inexact.

Some writers have proposed what are known as " Straight Line Formulæ " to take the place of those giving varieties of curves. Such formulæ are unscientific, and cannot possibly accord, with any degree of accuracy, with the results of experiments, or with true theoretical conditions, except over a very limited range. Others propose to use compound curves, consisting of a portion of Euler's curve, over the range of the higher values of $\frac{L}{r}$ coupled up to the direct compression value of the section at $\frac{L}{r} = O$ by other curves, such as parabolas, etc. Others, again, propose portions of Euler's curve, joined up to the extreme values by straight lines.

All these devices are purely empirical, and it would be, in reality, much better to plot the results of experiments in diagram form and draw curves, by freehand or other methods, passing through the loci of the apparently most representative results, and then obtain intermediate values by scaling or by construction of empirical formulæ.

This has been attempted several times, and it has then been made the practice to draw curves, not through the most frequently occurring loci or points, but as nearly as possible so as to pass through the lowest values, so that nearly all lie above the curves. Of course, if experiments could be made on all types and sizes of struts, such a method would be absolutely safe, and it would not even be necessary to allow any additional factors of safety, because, if struts were never constructed of proportions, such as would cause them to be represented by loci below the curves, there could never be any failures.

All such methods are very limited in their application, as if, for example, a new section was designed, there would be no guarantee that the results would not be different from those obtained from previous sections. Again, as such methods make no discrimination between various forms of section, but are simply based on the results of using the most inefficient forms, it follows that the methods must, on the average, result in considerable waste of material, and consequently unnecessary cost.

With the present tendency to increase the magnitude of stuctures, it becomes imperative to pay more attention to all details.

CHAPTER III.

MOMENTS OF INERTIA OF SECTIONS.

THE theories of Moments of Inertia and Radius of Gyration are well known generally, but having regard to their importance in connexion with the flexure of struts, it is necessary to consider some matters, not usually dealt with, as bearing on the question of the merits or efficiencies of various forms of cross section.

It has been shown that when a rod is bent, the intensities of the fibre stresses vary directly as the distance from the neutral plane. On a cross section of the beam, the intersection of the neutral plane forms a line at right angles to the direction of the bending, and that line is called the neutral axis of the section.

In Fig. 3, let B D be the cross section of such a beam. Any small portion of the area Δ A carries a stress proportional to its distance from the neutral axis XX. If p''_δ is the fibre stress intensity at the point farthest from the axis, $= \delta$, then the stress intensity on Δ A is $\frac{p''_\delta y}{\delta}$

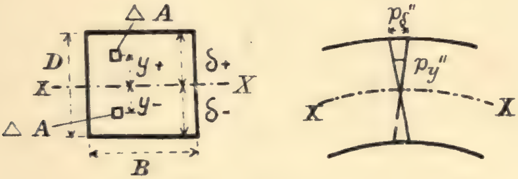


FIG. 3.

$$\text{The stress on } \Delta A = \frac{p''_\delta \, y}{\delta} \Delta A$$

$$\text{and its moment about } X X = \frac{p''_\delta \, y}{\delta} \Delta A \cdot y = \frac{p''_\delta \, y^2}{\delta} \Delta A$$

Adding together the moments of the stresses on the whole of the small areas into which the total area may be divided, the moment for the whole section, about X X

$$= \Sigma \frac{p''_\delta}{\delta} y^2 \cdot \Delta A$$

This expression is not strictly true, as each of the small areas, so long as it has any magnitude, can be divided into smaller portions, and the sum of the moments of the forces on these will not equal $\Sigma \frac{p''_\delta}{\delta} y^2 \cdot \Delta A$ because some of the subdivisions are farther from X X than others, and the mean of the squares of their distances will not equal the square of the mean distance y. If, however, ΔA is diminished in size, the error also diminishes, and on continuing to diminish ΔA towards its limit $d A$ the error finally disappears, and the result becomes exact.

The moment $\Sigma \frac{p''_\delta}{\delta} y^2 \cdot \Delta A$ then becomes $\frac{p''_\delta}{\delta} \int_{\delta-}^{\delta+} y^2 \cdot d A$ which is the total moment about X X of all the forces acting on the section. If this moment is divided by $\frac{p''_\delta}{\delta}$ the resulting value

$$\int_{\delta-}^{\delta+} y^2 \cdot d A = I_0 \qquad . \qquad . \quad (17)$$

is the sum of the areas of each of the parts of the whole area multiplied by the square of its distance from the neutral axis, and is known as the Moment of Inertia of the area, about X X. It will be seen that although the distances on one side of the axis are +, and on the other side −, when these are raised to the second power, or squared, they all become +.

When a rod is subjected to a bending moment, the moments of all the forces acting on any cross section must balance, and their algebraic sum = 0. If M is the moment of the external forces about X X of the section, then M must equal $\frac{p''_\delta}{\delta} I_0$ and if M is known, the extreme fibre stress intensity is

$$p''_\delta = \frac{M\,\delta}{I_0} \qquad . \qquad . \quad (18)$$

and the intensity on any other fibre

$$p''_y = \frac{M\,y}{I_0}$$

The Moment of Inertia is therefore a very important value.

There must be some distance, r, from the axis X X such that, if the whole area A were concentrated at that distance, Ar^2 would equal I_0. This distance is called the Radius of Gyration of the section, and from the above,

$$r^2 = \frac{I_0}{A} \quad \text{or} \quad r = \sqrt{\frac{I_0}{A}} \; . \qquad . \quad (19)$$

When calculating the stresses in beams subjected to bending, it is seldom required to know any of the fibre stresses except those at the extreme fibres = $p''_{\delta+}$ or $p''_{\delta-}$ which are the maximum. Sometimes, in using the equation (18), $p''_\delta = \frac{M\,\delta}{I_0}$ to avoid labour in repeatedly multiplying M by δ and then dividing by I_0 it is made a practice to calculate the value of $\frac{I_0}{\delta}$ and call it R or the moment of resistance of the section.

Then $$p''_\delta = \frac{M}{R}$$

The use of R is perfectly correct, so long as the distances δ_+ and δ_- of the extreme fibres on each side of the axis are the same. When this is not the case, two values of R have to be calculated, one $\frac{I_0}{\delta_+}$ and the other $\frac{I_0}{\delta_-}$. One of these values has to be used for stresses on one side of the axis, and the other for stresses on the other side. These values of R are termed the Modulus of the section, and when unequal, the Greatest Modulus, and Least Modulus, or other somewhat similar names. This is very likely to lead to confusion and mistakes, and it is much better to always use the general equation (18), $p''_\delta = \frac{M\,\delta}{I_0}$ when there can be no doubt about the results, and no liability to use wrong values.

It is well known that for a rectangle, referred to the neutral axis parallel to two of the sides, $I_0 = \frac{B\,D^3}{12}$ where B is

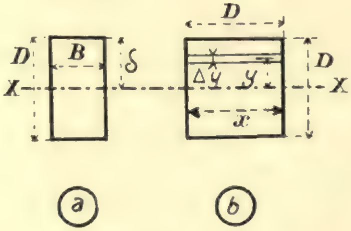


FIG. 4.

the dimension parallel to the axis, see Fig. 4 (*a*), and D the total depth at right angles to the axis, and when the rectangular figure is a square, B = D (Fig. 4 (*b*)), and $I_0 = \frac{D^4}{12}$

This is arrived at as follows :—

Consider any narrow strip of the area, parallel to the axis X X, and let y be its distance from the axis ; B its breadth ; and Δy its depth. Then the area of the strip is $B \cdot \Delta y$ and its Moment of Inertia about X X

$$= I_X = B \cdot \Delta y \times y^2 = B \cdot y^2 \cdot \Delta y$$

Now, if Δy be diminished without limit, this becomes $I_X = B \cdot y^2 \cdot dy$ and for the whole square the Moment of Inertia about X X

$$= I_0 = 2B \int_0^{\frac{D}{2}} y^2 \cdot dy$$

and as, in this case, B is a constant $= D$; and when $y = 0$ $I_X = 0$; the whole integral is

$$I_0 = \frac{D^4}{12} \qquad . \quad . \qquad (20)$$

If the square is placed diagonally, so that X X passes through two opposite corners, Fig. 5, the breadth of any strip parallel to X X is

$$x = \sqrt{2} \cdot D - 2 \cdot y$$

and its area

$$= (\sqrt{2} \cdot D - 2 \cdot y)\, \Delta y$$

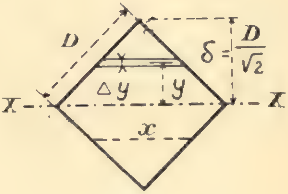


FIG. 5.

The Moment of Inertia about X X is

$$I_X = (\sqrt{2} \cdot D. - 2 \cdot y)\, \Delta y \times y^2 = (\sqrt{2} \cdot D \cdot y^2 - 2 \cdot y^3)\, \Delta y$$

and when Δy is diminished without limit, this becomes

$$I_x = (\sqrt{2} \cdot D \cdot y^2 - 2 \cdot y^3)\, dy$$

Then for the whole square

$$I_0 = 2 \int_0^{\frac{D}{\sqrt{2}}} (\sqrt{2} \cdot D \cdot y^2 - 2 \cdot y^3)\, dy$$

and as, in this case also, $I_0 = 0$ when $y = 0$

the whole integral

$$I_0 = \frac{D^4}{3} - \frac{D^4}{4} = \frac{D^4}{12} \quad . \quad (21)$$

From this we see that the Moment of Inertia of a square, about a neutral axis passing diagonally through two opposite corners, is the same as that about a neutral axis parallel to two of the sides.

If the beam, of square section, is subjected to a bending moment M, the extreme fibre stress intensity, $p'' = \frac{M\,\delta}{I_0}$ δ being the distance of the extreme fibre from the neutral axis.

If the plane of bending is parallel to the sides of the square

$$p'' = \frac{M}{I_0} \times \frac{D}{2} \quad . \quad . \quad (22)$$

If the bending takes place about a diagonal axis,

$$p'' = \frac{M}{I_0} \times \frac{D}{2} \cdot \sqrt{2} = \frac{M}{I_0} \times \frac{D}{\sqrt{2}} \quad . \quad (23)$$

Therefore, under the same bending moment, the beam would have to carry higher fibre stress intensities (in the ratio of $\frac{2}{\sqrt{2}}$) if placed diagonally than if placed with a side parallel to the plane of bending.

If, as in Fig. 6, when the square section is placed diagonally, small triangular portions are cut off both the upper and lower apexes, the moment of inertia will be reduced. The distance δ is, however, also reduced, and although the reduction of I_0 tends to increase p'', it will be found, that at first

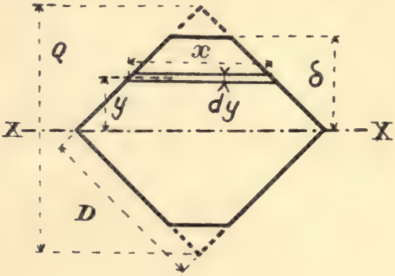


FIG. 6.

the reduction of δ more than compensates for the reduction of I_0 and although the sectional area has been cut down, the fibre stress intensity has been reduced, instead of being increased as might have been expected.

It is evident that a continued cutting away at the apexes cannot continue to reduce the fibre stress intensities without limit, because eventually the whole section would be cut away, and there would be nothing left to carry any stresses. It follows, therefore, that some point must be arrived at where the strength of the section remaining is a maximum.

Referring to Fig. 6, the width of a strip of the area parallel to X X is, as before, $x = \sqrt{2} \cdot D - 2y$ and its depth Δy which, on being diminished, becomes dy Then, as before,

$$I_{0y} = 2\int_0^y (\sqrt{2} \cdot D y^2 - 2y^3)\, dy$$

If, now, the section is cut down until the distance from X X to the extreme fibre is $y = \delta$ and it is subjected to a bending moment M then $p''_\delta = \frac{M\,\delta}{I_{0\delta}}$ and if p''_δ has a minimum value, it follows that, at the same time, $\frac{\delta}{I_{0\delta}}$ must be a minimum, or $\frac{I_{0\delta}}{\delta}$ a maximum.

Now, by integration,

$$I_{0y} = \frac{2 \cdot \sqrt{2} \cdot D \cdot y^3}{3} - y^4$$

and when y becomes δ

$$I_{0\delta} = \frac{2 \cdot \sqrt{2} \cdot D \cdot \delta^3}{3} - \delta^4$$

Then

$$\frac{I_{0\delta}}{\delta} = \frac{2 \cdot \sqrt{2} \cdot D \cdot \delta^2}{3} - \delta^3$$

and when $\frac{I_{0\delta}}{\delta}$ is a maximum, we have, by differentiation,

$$\frac{4 \cdot \sqrt{2} \cdot D \cdot \delta}{3} - 3\,\delta^2 = 0$$

Solving this equation, we find that (for min. p'')

$$\delta = \frac{4 \cdot \sqrt{2} \cdot D}{9} \quad . \quad . \quad (24)$$

and calling the diagonal of the square Q we have, by substitution,

$$\delta \text{ (for min. } p'') = \frac{4}{9}\,Q \quad . \quad (25)$$

or the whole depth of the section is now

$$2\delta = \frac{8}{9}\,Q \quad . \quad . \quad . \quad (26)$$

We then find that $I_{0\delta}$ which, for the altered section is I_0 becomes

$$I_0 = 2\int_0^\delta (\sqrt{2} \cdot D \cdot y^2 - 2y^3)\, dy$$

$$= 2\left(\frac{\sqrt{2} \cdot D \cdot \delta^3}{3} - \frac{\delta^4}{2}\right) = \frac{512}{2187} D^4 - \frac{1024}{6561} D^4$$

$$= \frac{512}{6561} D^4 \quad . \quad . \quad . \quad (27)$$

Then

$$p'' = \frac{M\,\delta}{I_0} = \frac{M \cdot 4 \cdot \sqrt{2} \cdot D \cdot 6561}{9 \cdot D^4 \cdot 512}$$

$$= \frac{\sqrt{2} \cdot M}{D^3} \times 5{\cdot}6953 \quad . \quad . \quad (28)$$

Now, for the whole square placed diagonally, we had, by (23),

$$p'' = \frac{\sqrt{2} \cdot M \cdot D}{2 \cdot I_0}$$

and by substituting the equivalent value of I_0 from (21), this becomes

$$p'' = \frac{\sqrt{2} \cdot M}{D^3} \times 6 \quad . \quad . \quad (29)$$

Therefore, under the same bending moment M, by cutting away the upper and lower apexes, until the over-all vertical dimension is $\frac{8}{9}$ of the original diagonal of the square, the intensity of the stress on the extreme fibres has been reduced in the ratio of

$$\frac{5{\cdot}6953}{6} \quad . \quad . \quad . \quad (30)$$

A consideration of the above shows that the *form* of the cross-section has a very great effect on the final result, and that, as stated, the use of a function such as that known as the Moment of Resistance of the section rather tends to hide such a fact.

It will be found, in many forms of cross section where the width or dimension parallel to the neutral axis diminishes as the distance from the axis increases, that the strength would be increased by removing portions of the material.

Further consideration leads to the conclusion that, if a considerable tapering off in the form of section decreases the strength, while, if a rectangular section is extended to any distance from the axis the strength continually increases, there must be some intermediate form where an increase would neither increase nor reduce the strength or stresses.

It also appears that it should be possible to design a cross section of such an outline that, no matter to what extent it was extended from the axis, or reduced, the Moment of Resistance would remain the same.

If, as before, X X is the neutral axis, y any distance from the axis, and x the breadth of the figure at the distance y then, taking the general expression for the moment of inertia of a figure about the neutral axis XX from (17) and substituting $x \cdot dy$ for dA

$$I_{Oy} = 2\int_0^y x\,y^2 \cdot dy$$

and the problem now is to find the value of x in terms of y

such that $p'' = \dfrac{My}{I_{Oy}}$ is constant,

or for any value of δ $\dfrac{I_{O\delta}}{\delta} = \dfrac{M}{p''}$

and if the section extends from X X to any distance ± y and not beyond, that value of $y = \delta$ and therefore

$$\frac{I_{0y}}{y} = \frac{I_{0\delta}}{\delta} = \frac{M}{p''} = \text{constant}$$

and

$$I_{0y} = \frac{M}{p''} \cdot y$$

Substituting this value of I_{0y} in (17), we have

$$\frac{M}{p''} \cdot y = 2 \int_0^y x\, y^2 \cdot dy$$

and, differentiating both sides

$$\frac{M}{p''} = 2\, x y^2$$

whence

$$x = \frac{M}{p''} \cdot \frac{1}{2 y^2} \quad . \quad . \quad . \quad (31)$$

and

$$y = \left(\frac{M}{p''} \cdot \frac{1}{2 x}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad (32)$$

For the purpose of delineating such a figure, definite values must be assigned to M and p''. Then, writing c for $\frac{M}{p''}$

$$I_{0y} = c y \quad \text{and} \quad I_{0\delta} = c\, \delta$$

whence

$$c = \frac{I_{0y}}{y} = \frac{I_{0\delta}}{\delta}$$

and, substituting these in (31) and (32), we have

$$x = \frac{c}{2 y^2} = \frac{I_{0y}}{2 y^3} \quad . \quad . \quad (33)$$

and

$$y = \left(\frac{c}{2 x}\right)^{\frac{1}{2}} = \left(\frac{I_{0y}}{2 x}\right)^{\frac{1}{3}} \quad . \quad (34)$$

Such a figure is drawn to scale in Fig. 7.

It consists of a curve having four branches, one in each of the quadrants formed by the intersection of the axes of X X and Y Y

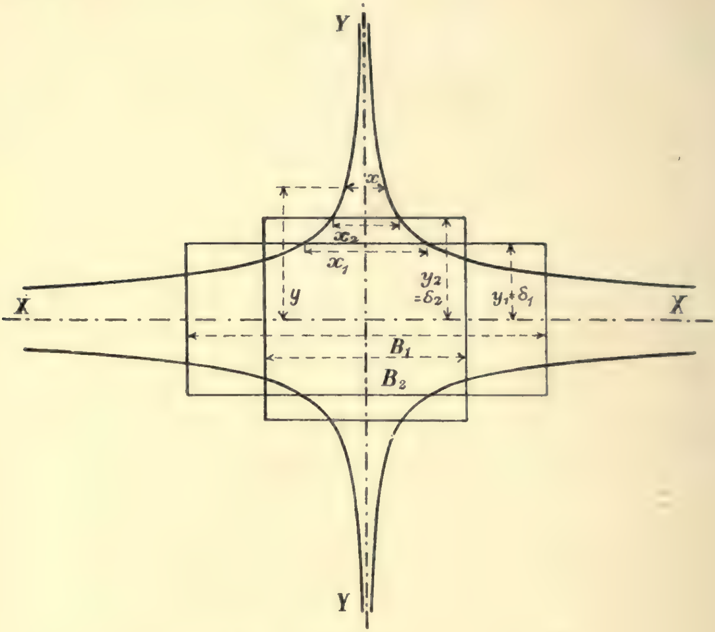


Fig. 7.

When y becomes ± 0 $x = \pm \dfrac{c}{2 \times 0} = \pm \infty$ and the branches of the curve gradually approach X X in both directions, X X being asymptotic to all four.

When $x = 0$ $y = \left(\dfrac{c}{2 \times 0}\right)^{\frac{1}{2}} = \pm \infty$ and the branches also approach Y Y in both directions,' and Y Y is asymptotic to all four.

If a beam could be constructed, having a cross section of this contour, and was subjected to a bending moment, we should have

$$p'' = \frac{M\delta}{I_{0\delta}} = \frac{M\delta}{c\delta} = \frac{M}{c}$$

that is, p'' would be constant whatever value might be given to δ between 0 and ∞. If the beam was made of any depth, such as 2δ $2\delta_2$ or generally 2δ the bending moment or load which it would carry would be the same, whatever depth it might be made, and although the sectional area and weight of the beam varied considerably.

The weight of the beam itself is to be considered as being included in the load producing the bending moment M, in the foregoing.

For a rectangle we have,

$$I_0 = \frac{B\,D^3}{12}$$

and, substituting for D its equivalent in terms of δ, this may be written

$$I_0 = \frac{2\,B\,\delta^3}{3}$$

where $$D = 2\,\delta$$

Then, substituting this value in (33), and writing δ for y we have

$$x = \frac{I_{0y}}{2y^3} = \frac{2 \cdot B \cdot \delta^3}{2 \cdot 3 \cdot \delta^3} = \frac{B}{3} \quad . \qquad (35)$$

Therefore, if we draw a number of rectangles as indicated in Fig. 7, of depths $2\,\delta_1$ $2\,\delta_2$ or of any depth $2\,\delta$ each having the same moment of inertia as that of the area included by the four branches of the curve between the limits $\pm\,\delta_1 \pm\,\delta_2$ or $\pm\,\delta$ it follows, from (35), that the widths B_1 B_2 etc. will always be

$$3\,x_1 \quad 3\,x_2 \text{ etc., or generally, } 3\,x \quad . \quad . \qquad (36)$$

Now consider one of these rectangles B D Fig. 8, the Moment of Resistance of B D is the same as that of the area enclosed by the dotted portions of the curved lines, to the distance δ from the axis X X. It was seen above that any extension of the limiting distance δ although it increased the area enclosed, left $\frac{I_0}{\delta}$ constant. Therefore, if any portions of the areas enclosed between the curved lines, outside of the rectangle, be added to the rectangle, equal portions being added above and below, such as to the limit δ_2 although the area of the figure is increased, $\frac{I_0}{\delta}$ again remains the same.

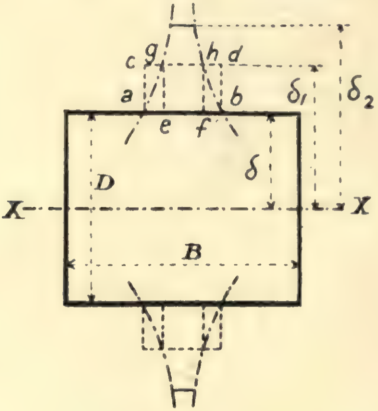


FIG. 8.

If the figure represented the cross-section of a beam, subjected to a bending moment M no matter to what extent it was extended, above and below equally, by adding portions bounded by the curves, p'' would remain constant.

If rectangular pieces were added to the original rectangle, above and below, such as *a b c d* these, being outside of the curved lines, would strengthen the section to resist bending. On the other hand, if rectangles such as *e f g h* lying within the curved lines, were added, the section would be weakened, although the area would be increased. The width of the rectangle *a b c d* is *a b* and is equal to x_δ or $\frac{B}{3}$ and it therefore appears that a sudden reduction below that limit introduces a weak point.

This investigation shows that, in designing cross-sections, any sudden or rapid reductions in dimensions as distance from the axis increases are to be avoided. In sections for columns it is not unusual to find light stiffening angles, or ribs, placed on the outside of comparatively heavy sections, and if these are within certain limiting dimensions they weaken the column, although not to the same extent as the section would be weakened if it was intended for a beam, because in the column the sectional area of these projecting parts assists, to some extent, in carrying the direct compressive stresses.

It can easily be found, in any design, if such projecting parts are within the limits or not. In Fig. 9, the section shown is not an unusual one for a column. It consists of a rolled **H** beam, with two channels riveted to its flanges, and placed with the flanges of the channels pointing outwards away from the axis. Or, instead of channels, plates might be riveted to the **H** beam, and the edges of the plates stiffened by small angles on the outer sides.

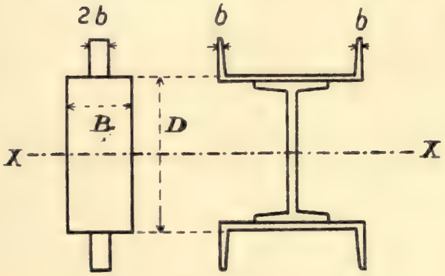


FIG. 9.

The Moment of Inertia I_{0_D} should be calculated for the portion of the section extending to the limit D, that is, including the webs of the channels, but not the projecting flanges; and the breadth B of the equivalent rectangle should be determined, having the depth D and the Moment of Inertia I_0 equal to those of the portion of the section under consideration. Therefore, as in a rectangle,

$$I_0 = \frac{B D^3}{12} \;; \quad B = \frac{12 I_0}{D^3} \qquad . \quad (37)$$

If, now, it is found that $2b$ or the sum of the breadths of the projecting flanges is less than $\frac{B}{3}$ it follows, from (35), that, so far as bending is concerned, they are a source of weakness.

In such a case it is very probable that, if the channels were reversed and the flanges made to point inwards, or if the stiffening angles along the edges of a flange plate were placed on the inside instead of the outside, a stronger column would be the result.

For example, with a section such as shown in Fig. 24 (*d*), it is very usual to place two channels at such a distance apart that the Moment of Inertia is the same in both directions. If we take two channels 12 inches by 3·5 inches by 33 lbs. per foot, the greatest Moment of Inertia of each is 191 inches^4 and the combined Moment $=$ 382 inches^4. If the channels are placed with their flanges pointing outwards, and so as to have the same Moment of Inertia, it will be found that they must be at such a distance apart that the over-all dimension in that direction is 13·94 inches, and the distance δ is then 6·97 inches. In that case the Moment of Resistance to bending, for unit stress intensity, is

$$\frac{382}{6 \cdot 97} = 55 \text{ inches}^3.$$

If the channels are placed with their flanges pointing inwards, the over-all dimension will be 10·42 inches, and $\delta = 5 \cdot 21$ inches. The Moment of Resistance is now 73·3 inches^3, or about 33 per cent. greater than before.

At the same time, if the flanges of the channels are turned inwards, the Radius of Gyration of the section, about X X, is reduced, and this is dealt with later.

There are many tables of Moments of Inertia of regular figures and of sections of bars and beams commonly used, such as angles, tees, H-beams, round, square, and rectangular sections, etc. It is not difficult to calculate the Moment of

Inertia of any irregular figure which can be divided into rectangles or triangles, but if any of the bounding lines are curved, each case requires special treatment. The Neutral Axis, for equal values of E in tension and compression, always passes through the centre of gravity of the figure.

We have seen that I_0 for a rectangle, about its neutral axis parallel to a side, is $\frac{B\,D^3}{12}$ If I is required about an axis coinciding with one side, Fig. 10, we have as before,

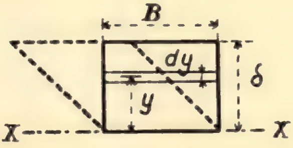


FIG. 10.

the area of a strip parallel to X X = B · dy and its Moment of Inertia about X X = $By^2 \cdot dy$ Then for the whole rectangle,

$$I_X = \int_0^\delta By^2 \cdot dy = \frac{B\,\delta^3}{3} \qquad . \quad (38)$$

The Moment of Inertia for any rectangle, of the depth δ, as indicated by dotted lines, will be the same.

For a triangle referred to its base, Fig. 11, let x = the length of a strip parallel to X X and y = its distance from

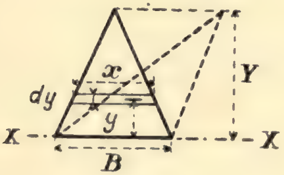


FIG. 11.

X X. Let Y be the perpendicular distance from the axis to the apex of the triangle and B = the base. Then

$x = B - y \times$ constant $= B - cy$ then, when $y = Y$ $x = 0$, and $0 = B - cY$ $\therefore$ $c = \frac{B}{Y}$ and $x = B - \frac{B}{Y}y = B\left(1 - \frac{y}{Y}\right)$

The area of the strip is $B\left(1 - \frac{y}{Y}\right)dy$, and its Moment of Inertia about X X $= B\left(1 - \frac{y}{Y}\right)dy \times y^2 = B\left(y^2 - \frac{y^3}{Y}\right)dy$

Then, for the whole triangle, $I_X = \int_0^Y B\left(y^2 - \frac{y^3}{Y}\right)dy$

the general integral is

$$B\left(\frac{y^3}{3} - \frac{y^4}{4Y}\right) + c$$

and as, $I_X = 0$ when $y = 0$

$$\therefore \quad c = 0$$

and

$$I_X = B\left(\frac{Y^3}{3} - \frac{Y^4}{4Y}\right) = \frac{BY^3}{12} \qquad (39)$$

The Moment of Inertia for any triangle on the base B and with perpendicular height Y is the same.

For a triangle referred to an axis parallel to its base and passing through its apex, Fig. 12, $x = y \times$ constant $= cy$

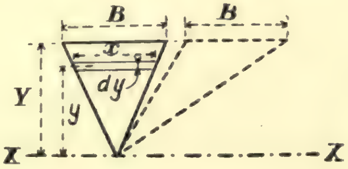


FIG. 12.

$$\therefore \quad B = cY \quad \text{and} \quad c = \frac{B}{Y} \qquad \therefore \quad x = \frac{B}{Y}y$$

The area of a strip parallel to X X $= x \cdot dy = \frac{B}{Y} y \cdot dy$

and its Moment of Inertia about X X $= \frac{B}{Y} y^3 \cdot dy$

For the whole triangle

$$I_X = \frac{B}{Y}\int_0^Y y^3 \cdot dy = \frac{B}{Y} \cdot \frac{Y^4}{4} = \frac{B\,Y^3}{4} \quad . \quad (40)$$

Here, again, I_X is the same for any triangle of base B and perpendicular height Y.

It is evident that the triangle in Fig. 12 is equal to the difference between the rectangle, Fig. 10, and the triangle, Fig. 11. Its Moment of Inertia about XX should then be the difference between those of the two figures. Therefore, writing Y for δ in (38) we have

$$I_X = \frac{B\,Y^3}{3} - \frac{B\,Y^3}{12} = \frac{B\,Y^3}{4} \qquad \text{as in (40)}$$

The Moments of Inertia of most of the ordinary sections can be calculated by these formulæ. For example, the regular hexagon, Fig. 13, has a Moment of Inertia, about axis

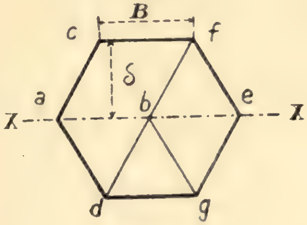


FIG. 13.

XX clearly equal to the sum of the moments, about that axis, of two rectangles = *acfb* and two triangles *bfe* as indicated in the upper half of the figure. Then, if the side of the hexagon is B, we have, from (38) and (39),

$$I_0 = 2\frac{B\,\delta^3}{3} + 2\frac{B\,\delta^3}{12} = \frac{5}{6}B\,\delta^3 \quad . \quad (41)$$

Alternatively, the hexagon could, as indicated in the lower half of the figure, be divided into six similar triangles, four of them with their bases coinciding with X X and two with their apexes touching X X Then, from (39) and (40), we should have

$$I_0 = 4\frac{B\,\delta^3}{12} + 2\frac{B\,\delta^3}{3} = \frac{5}{6}B\,\delta^3$$

the same as (41).

In Fig. 14 (*a*), I_0 for the two rectangles *a b c d* + *e f g h* is,

I_0 (rectangle *a b g h*) − I_0 (rectangle *c d e f*)

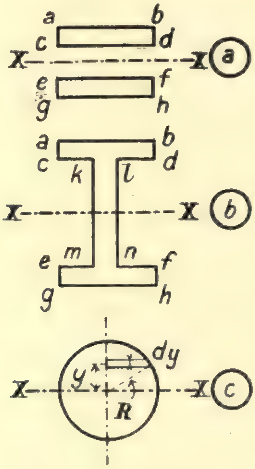


FIG. 14.

In Fig. 14 (*b*), I_0 for the web and double flange is,

I_0 (rectangle *a b g h*) − $2I_0$ (rectangle *c k e m*)

For a circle, referred to a diameter, Fig. 14 (*c*),

$$x^2 = R^2 - y^2 \qquad \therefore x = (R^2 - y^2)^{\frac{1}{2}}$$

then, taking one quadrant,

$$I_X = \int_0^R xy^2 \cdot dy = \int_0^R (R^2 - y^2)^{\frac{1}{2}} \cdot y^2 \cdot dy$$

and, for the whole circle

$$I_0 = 4\int_0^R (R^2 - y^2)^{\frac{1}{2}} \cdot y^2 \cdot dy$$

which by integration, gives

$$I_0 = \frac{\pi}{4} R^4 \quad . \quad . \quad . \quad (42)$$

If required to find the Moment of Inertia of any figure about an axis not passing through its centre of gravity, as in Fig. 15, where B D is a rectangle, and X X the neutral axis, then $I_0 = \frac{B D^3}{12}$ and it can be shown that the Moment of Inertia I_{X_1} about any other axis $X_1 X_1$ parallel to X X is $I_0 + B D \times Z^2$ that is I_{X_1} is equal to I_0 plus the area of the figure, multiplied by the square of the distance between the neutral axis and the new axis. This applies, not only to a rectangle, but to any figure.

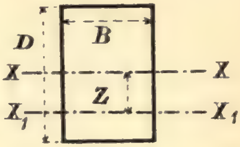


FIG. 15.

It is well known that for every figure in which I_0 is not the same about all neutral axes there is a pair of neutral axes, at right angles to each other, about one of which I_0 is a maximum, and about the other I_0 is a minimum for the whole figure. These two axes are called the principal axes. It is also known that if an ellipse be constructed, so that the square of the semi-axis major = I_0 max. and the square of the semi-axis minor = I_0 min. as in Fig. 16, where $\overline{OY}^2 = I_0$ max. and $\overline{OX}^2 = I_0$ min. and placed so that X X is parallel to the axis of the figure, about which I_0 is greatest ; if a straight line be drawn through O such as O X_1 parallel to any other axis through the centre of gravity of the figure, and if a perpendicular p be drawn to O X_1 from the most distant point in the ellipse, then $p^2 = \overline{OT}^2$ represents I_0 about that axis.

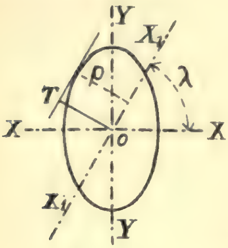


FIG. 16.

From the properties of the ellipse

$$\overline{OT}^2 = \overline{OY}^2 \cdot \cos^2\lambda + \overline{OX}^2 \cdot \sin^2\lambda$$

Therefore, the Moment of Inertia about any axis $X_1 X_1$

$$= I_{OX_1} = I_{OX} \cdot \cos^2\lambda + I_{OY} \cdot \sin^2\lambda \qquad (43)$$

if λ is the angle between the axes X X and $X_1 X_1$

It follows that, for any figure, if I_0 about each of the principal axes is the same, the two axes of the ellipse are equal, and the eccentricity nil ; therefore the ellipse becomes a circle and I_0 is constant about any axis passing through the centre of gravity of the figure . . (44)

It can be demonstrated, but should be apparent without demonstration, that if I_0 is the same about one of the principal axes and about any other neutral axis, not at right angles to it, then the ellipse again becomes a circle, and I_0 must be the same about any neutral axis.

It should also be apparent, but not so clearly, from a study of Fig. 16, and the text, and it can be demonstrated, that if the figure is symmetrical about any two neutral axes, not at right angles to each other, although the disposition of its parts may not be the same about each of these two axes I_0 is again the same about any neutral axis . . (45)

When the Moment of Inertia is the same about all neutral axes the figure is said to be "Isotropic."

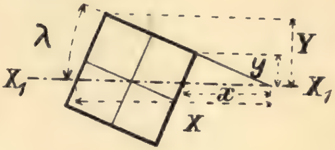


FIG. 17.

As a practical example, consider a square, Fig. 17. As the disposition of the figure is the same about each of the axes parallel to the sides, by (44), I_0 should be the same about any neutral axis. We saw, by (21), that this was the case about axes passing through the corners, and parallel to the sides. Now take any other neutral axis $X_1 X_1$ Fig. 17. By (39) we have, for the whole square

$$I_0 = 2\left(\frac{X \cdot Y^3}{12} - \frac{x \cdot y^3}{12}\right) = \frac{1}{6}(X \cdot Y^3 - x \cdot y^3)$$

Now

$$Y = \frac{D}{2}\cos\lambda + \frac{D}{2}\sin\lambda = \frac{D}{2}(\cos\lambda + \sin\lambda)$$

$$y = \left(\frac{D}{2} - \frac{D}{2}\tan\lambda\right)\cos\lambda = \frac{D}{2}(\cos\lambda - \sin\lambda)$$

$$x = \frac{\frac{D}{2} - \frac{D}{2}\tan\lambda}{\sin\lambda} = \frac{D}{2}\left(\frac{1}{\sin\lambda} - \frac{1}{\cos\lambda}\right)$$

$$X = \frac{D}{\cos\lambda} + \frac{D}{2}\left(\frac{1}{\sin\lambda} - \frac{1}{\cos\lambda}\right) = \frac{D}{2}\left(\frac{1}{\sin\lambda} + \frac{1}{\cos\lambda}\right)$$

$$\therefore I_0 = \frac{1}{6}\left\{\frac{D}{2}\left(\frac{1}{\sin\lambda} + \frac{1}{\cos\lambda}\right) \times \frac{D^3}{8}(\cos\lambda + \sin\lambda)^3\right.$$

$$\left. - \frac{D}{2}\left(\frac{1}{\sin\lambda} - \frac{1}{\cos\lambda}\right) \times \frac{D^3}{8}(\cos\lambda - \sin\lambda)^3\right\}$$

Which reduces to

$$\frac{D^4}{12}(\sin^2\lambda + \cos^2\lambda)$$

and, as $\sin^2\lambda + \cos^2\lambda = 1 \quad \therefore I_0 = \frac{D^4}{12}$

and this being true for any value of λ and the same value as in (20) and (21), therefore I_0 is the same about any neutral axis (Q. E. D.).

The same result would be arrived at by the application of (45). The square is symmetrical about X X in Fig. 4; it is also symmetrical about X X in Fig. 5, although not similarly disposed about each of them. These axes are at 45° with each other, that is, not at a right angle, and therefore I_0 is the same about any neutral axis.

It follows, from the preceding, that I_0 is the same about any neutral axis, in any regular figure, such as an equilateral triangle, square, pentagon, etc., and any figure of a star-like form, with any number of rays or points, of course greater than two, whether odd or even, and whether bounded by straight or curved lines, and symmetrically arranged. All such figures are isotropic.

In investigating the strength of struts, these facts are of the greatest importance, because, in nearly all cases, the strut is free to bend in any direction.

If the modulus of elasticity is not the same in tension as in compression, the neutral axis with respect to the internal stresses, at a section, will not pass through the centre of gravity of the section. For a given maximum stress intensity, the moment of the stresses on one side of the axis will not be equal to that of the stresses on the other side.

In Fig. 18, let XX be the axis of a beam, under a bending moment. If the section is rectangular, the breadth B will be constant, and the distributions of the strains over the section will be represented by the two triangles *o a c* and *o b d*. Let E_t in tension be greater than E_c in compression; then a given stress intensity will produce a less strain on the tension side than on the compression side, and the smaller triangle *o b d* will represent the tensile strains, while the larger triangle *o a c* will represent those in compression. The neutral axis will be between the tension side and the mid depth of the section.

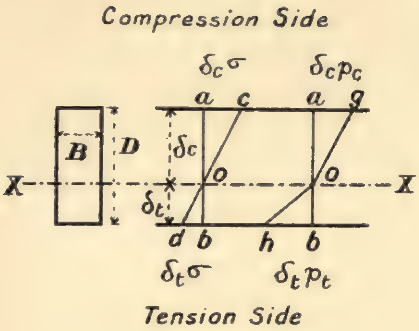


FIG. 18.

As stress intensities are taken as varying directly as the corresponding strains, the distribution of the stress intensities over the section will also be represented by two triangles *o a g* and *o b h*, and the triangle on the tension side will have a more obtuse angle at its apex, o, than the other.

As the moment of the whole of the internal stresses must equal the bending moment, the sum of the compressive stresses along with that of the tensile stresses must form a couple, and therefore the area of the triangle *o a g* must equal the area of *o b h*. We then have the following:—

Let σ be the strain at unit distance from the axis. Then

Strain at distance δ_c from the axis $= \delta_c \sigma$

Strain at distance δ_t from the axis $= \delta_t \sigma$

Intensity of compressive stress at distance $\delta_c = \delta_c p_c$
$= \delta_c \sigma \mathrm{E}_c$

Intensity of tensile stress at distance $\delta_t = \delta_t p_t = \delta_t \sigma \mathrm{E}_t$

Then

$$\text{Area } oag = \frac{\delta_c \times \delta_c p_c}{2} \quad \text{and area } obh = \frac{\delta_t \times \delta_t p_t}{2}$$

Therefore, as the two areas are equal, we have

$$\delta_c^2 p_c = \delta_t^2 p_t$$

$$\therefore \quad \delta_c^2 E_c = \delta_t^2 E_t$$

and, substituting $D - \delta_c$ for δ_t

$$\delta_c^2 E_c = (D - \delta_c)^2 E_t$$

$$\therefore \quad \delta_c = (D - \delta_c)\sqrt{\frac{E_t}{E_c}}$$

$$\therefore \quad \delta_c\left(1 + \sqrt{\frac{E_t}{E_c}}\right) = D\sqrt{\frac{E_t}{E_c}}$$

$$\therefore \quad \delta_c = D\frac{\sqrt{\frac{E_t}{E_c}}}{1 + \sqrt{\frac{E_t}{E_c}}} \quad . \quad . \quad . \quad (46)$$

or, if we call the ratio $\frac{E_t}{E_c} = q$

$$\delta_c = D\frac{\sqrt{q}}{1 + \sqrt{q}} \quad . \quad . \quad . \quad (47)$$

For a given maximum stress intensity in compression, a beam will carry a greater bending moment if E_t is greater than E_c On the other hand, it would carry a less bending moment if E_t were less than E_c

If the section is not rectangular, the neutral axis will not divide the depth in the same proportion, but the distribution of stress intensity will vary in the same manner.

This question of unequal values of E_t and E_c is of more importance in beams than in struts. In beams the stresses on one side of the axis are always compressive and on the other side tensile, while in a strut there is no tension until the maximum bending stress intensity is greater than that of the direct stress, as described in connection with Fig. 50.

So long as the stresses on the section of a strut are wholly compressive, differences between E_t and E_c have no effect, because the result of greater or less bending is then merely to increase or reduce the amount of compression, and the strains are then governed only by the value of E_c When the maximum intensity of the bending stresses is greater than the direct, tension appears, but unless the bending stresses are relatively great the tension area, as in Fig. 50, is small and its effect on the position of the neutral axis, etc., is also small.

If the section is not a simple one, but is made up of a number of parts which cannot conveniently be divided into a few simple figures, much time may be saved by finding the position of the neutral axis and the Moment of Inertia by graphic methods. Such methods are described in many good text books.

CHAPTER IV.

PROPERTIES OF VARIOUS COLUMN SECTIONS.

IT has been recognized by most experimenters on the strength of struts that results vary with the form of the cross section, as well as with the sectional area and radius of gyration. All authorities state that the reasons for the variation with the form, and the laws governing the same, are obscure and have never been determined. No satisfactory investigation of the question appears to have been made, and no existing formulæ make allowance for such variation, except by the introduction of certain empirical constants.

Some writers attribute the whole of the variations to stresses due to local flexure and distortions, and in the case of built up sections, formed of plates, angles, etc., where the thicknesses of parts are comparatively small, the want of local stiffness is undoubtedly a considerable factor, but does not explain the most important variations. For example, the want of local stiffness cannot be held to explain any difference in the strengths of solid columns of square sections as compared with those having sections of circular form, and yet there are considerable differences.

All the formulæ for design, which have been deduced from theoretical considerations, make use of the area of section and the radius of gyration, without considering the general dimensions. It has been suggested to modify these formulæ in many ways to allow for local flexure, and attempts have been made to introduce empirical constants of varying values for square, circular, angle, H, and other forms of section, but without very satisfactory results.

Much money has been spent on experiments, but no definite results have been obtained, and owing to the subject being obscure, the observations which have been made and recorded during these experiments have not been taken in such a way as to give data complete enough for a thorough investigation.

Consider a strut of considerable length, so that the stresses due to main flexure are of importance. If the section was a square, it would be found that the usual formulæ for calculating the strength or carrying capacity indicate an equal strength for any direction of flexure. For example, as only the length L and the radius of gyration r of the section are considered, it would appear that the ultimate load P which the strut would carry would be the same if deflection took place in the direction of one of the diagonals as if it took place in a direction parallel to two of the sides. If, now, the strut was constrained to deflect in the direction of a diagonal, and small portions were removed, as in Fig. 6, it would appear that the ultimate strength of the strut would be increased, although not to the same extent as in the case of a beam, because, as the parts were cut away, the intensity of the direct stress would increase, owing to the reduction of area, and the point of maximum strength would be reached when a less amount had been removed.

This shows that the form of the section has a great effect on the final result, and suggests one direction for research.

When a strut fails under the main stresses, the maximum intensity is the sum of the direct stress intensity and the maximum intensity caused by the bending moment. This bending moment M equals Pv, if P is the total load, and v the deflection, or what is the same thing, the deviation of the line of application of the load P from that neutral axis of the section which is normal to the plane of bending.

Confining the investigation, in the first place, to sections for which I_0 is constant about any neutral axis, and which are termed Isotropic sections, take, first of all, a circular section. For an ultimate load P let p'' be the difference between the direct stress intensity p' and the ultimate intensity f Then

p'' is the maximum fibre stress intensity possible under the bending action. Now δ is, in this case, the radius of the circle;

$$\text{therefore} \quad p'' = \frac{P\,v\,\delta}{I_0} \quad \text{and} \quad v = \frac{p''\,I_0}{P\,\delta} \qquad (48)$$

where v is the *maximum permissible deflection* or deviation of centre of pressure from the neutral axis to keep f and p'' within their limits.

As the circular section is symmetrical, and similar about all neutral axes, the permissible deviation v is the same in all directions, and the loci of all the points at distance v if plotted on the circular section will form another circle with radius v concentric with the circle bounding the section. Thus, in Fig. 19, the larger circle, of radius δ or R represents the section, and the smaller circle, with radius v represents an area, within which the centre of pressure must lie to keep the maximum stress intensity within the assigned limit.

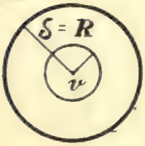


FIG. 19.

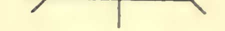


FIG. 20.

Now take a square section, Fig. 20. Here, again, by (44), I_0 is constant about any neutral axis, and by (48),

$$v = \frac{p''\,I_0}{P\,\delta}$$

but as δ is not constant, it follows that v must be a variable dimension. If deflection takes place parallel to two of the sides, then

$$\delta = \frac{D}{2} \quad \text{and} \quad v = \frac{2\,p''\,I_0}{P\,D}\,. \qquad (49)$$

These values of v plotted, to a suitable scale, on the figure give the four points *a a a a*.

If deflection takes place along a diagonal,

$$\delta = \frac{\sqrt{2}\,D}{2} = \frac{D}{\sqrt{2}} \quad \text{and} \quad v = \frac{\sqrt{2}\,p''\,I_0}{P\,D} \qquad (50)$$

Plotted to the same scale, along the diagonals, this gives the four points $b\,b\,b\,b$. It will be seen that, as v along the diagonal is to v parallel to the sides, as $\sqrt{2}$ is to 2, the points b lie on straight lines between the points a

For any intermediate direction of deflection (such as $O\,X_1$) δ_{X_1} is the distance from O to the line T T drawn through the corner of the square, perpendicular to $O\,X_1$ For this direction, the resulting value of v gives a point c which, as will be shown, lies in a straight line between a and b The loci of all the extreme points of the value v lie on lines forming a square, concentric with the square bounding the section, but turned with its diagonals at an angle of 45° with the diagonals of the main square. To keep the maximum fibre stress intensity within the assigned limit, the centre of pressure must lie within the smaller square, in the figure.

The proof that, as stated in the previous paragraph, the locus of the points c is a straight line at right angles to the diagonal of the main square, is as follows :—

In Fig. 21, O A B is a right-angled triangle, the angle B O A being ϕ $\overline{OT}$ is any line drawn through O at an angle θ with $\overline{OB}$ and $\overline{BT}$ is a line passing through B and perpendicular to $\overline{OT}$ D is any point in $\overline{OA}$ taken so that $\overline{OD} \times \overline{OA} = U$ which is then a constant. Let c be a point in $\overline{OT}$ taken so that

$$\overline{OC} \times \overline{OT} = U$$

Then the line $\overline{DF}$ is the locus of the points c as the angle varies from 0 to ϕ From c drop a perpendicular y to the line $\overline{OD}$ and let x be the distance between D and the foot of y.

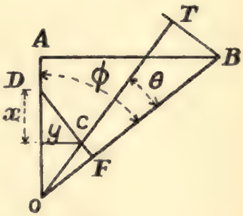


FIG. 21.

By hypothesis,

$$\overline{OD} \times \overline{OA} = \overline{OC} \times \overline{OT} = U$$

Now,

$$\overline{OB} = \frac{\overline{OA}}{\cos\phi} \text{ and } \overline{OT} = \overline{OB} \cdot \cos\theta = \overline{OA}\frac{\cos\theta}{\cos\phi}$$

therefore

$$\overline{OC} = \frac{\overline{OD} \times \overline{OA}}{\overline{OT}} = \overline{OD}\frac{\cos\phi}{\cos\theta}$$

Then

$$y = \overline{OC} \cdot \sin(\phi - \theta) = \overline{OD}\frac{\cos\phi}{\cos\theta}\sin(\phi - \theta)$$

$$= \overline{OD}\frac{\cos\phi}{\cos\theta}(\sin\phi \cdot \cos\theta - \sin\theta \cdot \cos\phi)$$

Or,

$$y = \overline{OD} \cdot \cos\phi\left(\sin\phi - \frac{\sin\theta}{\cos\theta}\cos\phi\right) \quad . \quad (51)$$

Again,

$$x = \overline{OD} - \overline{OC} \cdot \cos(\phi - \theta) = \overline{OD} - \overline{OD}\frac{\cos\phi}{\cos\theta}\cos(\phi - \theta)$$

$$= \overline{OD}\left(1 - \cos^2\phi - \sin\phi\frac{\sin\theta}{\cos\theta}\cos\phi\right)$$

therefore

$$x = \overline{OD} \cdot \sin\phi\left(\sin\phi - \frac{\sin\theta}{\cos\theta}\cos\phi\right) \quad . \quad (52)$$

Dividing (51) by (52) we have

$$\frac{y}{x} = \frac{\cos\phi}{\sin\phi} = \text{cotan}\,\phi$$

therefore $y = x \cdot \text{cotan}\,\phi$ and this is an equation to a straight line.

Then, as y also $= x \cdot \tan(\mathrm{ODF})$

$$\tan(\mathrm{ODF}) = \operatorname{cotan}\phi$$

and the triangle O D F is right angled, the angle O F D being the right angle. Therefore $\overline{\mathrm{DF}}$ is a straight line at right angles to $\overline{\mathrm{OB}}$ (Q.E.D.) . . . . . (53)

Comparing the above with Fig. 20, $\overline{\mathrm{OC}}$ corresponds with v, $\overline{\mathrm{OT}}$ with δ, and U with the value $\frac{p'' \mathrm{I}_0}{\mathrm{P}}$ and it will be seen that the equations apply to a portion of a square or to a portion of any other figure, so long as O A B is a right angle and $\frac{p'' \mathrm{I}_0}{\mathrm{P}}$ is constant.

For any figure bounded by straight lines and having I_0 constant, it is a very easy matter to draw the figure representing the area or field limiting the centre of pressure. From the centre of gravity of the figure draw straight lines radiating to each of the angles in the perimeter. Also draw radiating straight lines perpendicular to all the sides and intersecting them, or to the lines of the sides produced if necessary. Next calculate the value of v, for any one position, and plot the corresponding point on the section; then, through that point, draw a straight line perpendicular to the adjoining vector through one of the angles of the perimeter; produce the line until it intersects the next vector, drawn perpendicular to the next side of the perimeter. From that point of intersection draw another line perpendicular to the next vector line, drawn to an angle of the perimeter, and produce it until it intersects the next vector line. Continue in this way until the figure closes, as it should do if drawn correctly.

For example, in Fig. 22, the figure of the section is an equilateral triangle. It is isotropic and I_0 is constant about any axis through O. From O draw radial lines through the angles. Draw other lines from O perpendicular to the sides, and intersecting them at d, e and f. Take any neutral axis, say XX parallel to ab. Calculate the value of $v = \frac{p'' I_0}{P \delta}$ from (48) or (129) taking $\overline{oe}$ as δ Mark the point g so that $\overline{og} = v$ From g draw a line perpendicular to $\overline{ob}$ and produce it to intersect $\overline{of}$ in h From h draw a second line perpendicular to $\overline{oc}$, and produce it to intersect $\overline{od}$ in k From k draw a third line perpendicular to $\overline{oa}$ and, produced, it should pass through the starting point g forming a closed figure ghk This figure, ghk is the area within which the centre of pressure must lie to keep f within the assigned limit.

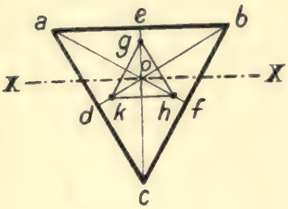


FIG. 22.

The same procedure can be adopted for any such figure, as for example in Fig. 23, and it will be found that, in all cases, where the perimeter forms a regular figure, the boundary of the pressure field will be of a similar form and turned with its corners pointing to the centres of the sides of the main figure. The figure of the pressure field is always of a form reciprocal to that of the main figure, whether regular or not, so long as I_0 is constant.

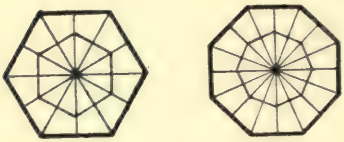


FIG. 23.

If all the angles of the main figure, measured internally, are less than 180°, the foregoing is complete. If, however, any of these angles are greater than 180°, as for example in the star shaped figure in Fig. 24 (*a*), in such a case it is only necessary to join up the projecting points by straight lines, to form an enclosing figure, and then draw the reciprocal figure corresponding to it. Thus, in Fig. 24 (*a*), the enclosing figure is a square and the reciprocal figure also a square.

In Fig. 24, (*b*) (*c*) (*d*) are illustrations of three sections commonly used, showing the form taken by the reciprocal figures. In (*c*) we have the outline of what is known as a Phoenix Column section. If the joint flanges do not project far enough to allow lines joining them to pass clear of the circular parts, it is necessary to draw *a b* and *d c* tangentially to the circle. Then the portions of the reciprocal figure, corresponding to the arcs *b c*, are other circular arcs concentric with them. The remainder of the reciprocal figure is constructed as before.

In Fig. 24, (*d*) is a section composed of two channels, placed at such a distance apart that I_0 is constant, about any neutral axis. If the dimension B exceeds D the reciprocal figure will take a diamond shaped form, as indicated.

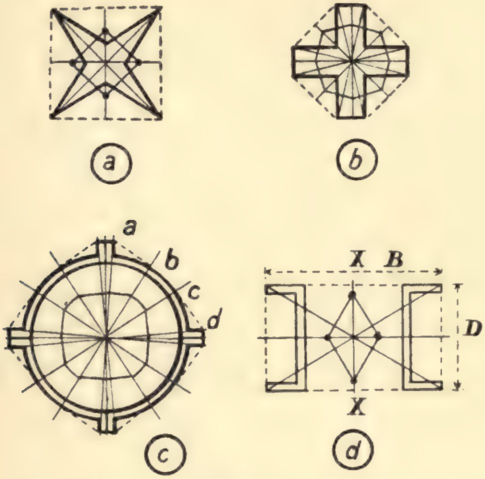


Fig. 24.

In all cases it will be seen that as the distance δ of the extreme fibre from the neutral axis increases, the permissible deviation v of P decreases, and where δ is a maximum, v is a minimum, and vice versa.

Even if a strut could be constructed with its axis perfectly straight, and the material be perfectly homogeneous throughout, and if the load could be applied directly along the axis, this consideration of the permissible deviation of P or the deflection, would be of some account, if the exact equations were used, as P could then only deviate from the axis by a deflection of the strut, and the direction of bending would affect the results, although to a very small extent. An examination of the equation (48) or (129) will show that if I_0 is constant, the maximum fibre stress intensity caused by a given load will, even under ideal conditions, vary with different directions of deflection, if δ varies.

If the section of the strut was, for example, a solid square and deflection took place parallel to the sides, and if p'' is, again, the difference between the direct and the ultimate permissible stress intensities, that is, the allowable stress intensity under bending action, the radius of curvature ρ to which the strut could be bent is, by (78),

$$\frac{1}{\rho} = \frac{M}{E\,I_0} = \frac{P\,v}{E\,I_0} \quad \text{wherefore} \quad \rho\,v = \frac{E\,I_0}{P} \qquad (54)$$

and, again,

$$\frac{1}{\rho} = \frac{p''}{E\,\delta} \quad \text{therefore} \quad \rho = \frac{E\,\delta}{p''} = \frac{E}{p''}\cdot\frac{D}{2} \qquad (55)$$

where D is the length of the side of the square.

Now, if the bending takes place along a diagonal,

$$\frac{1}{\rho} = \frac{P\,v}{E\,I_0} \quad \text{and, as before,} \quad \rho\,v = \frac{E\,I_0}{P} \qquad (56)$$

and

$$\frac{1}{\rho} = \frac{p''}{E\,\delta}$$

but as δ is now half the diagonal of the square

$$= \frac{D}{2}\cdot\sqrt{2} = \frac{D}{\sqrt{2}}$$

we have

$$\frac{1}{\rho} = \frac{p''\sqrt{2}}{E\,D} \quad \text{therefore} \quad \rho = \frac{E}{p''}\cdot\frac{D}{\sqrt{2}} \quad . \qquad (57)$$

In both cases $\rho\, v = \frac{E\, I_0}{P}$ and therefore, for a given load, $\rho\, v$ is a constant for any direction of bending, but ρ varies directly as δ and therefore v varies inversely as δ . (58)

With a gradually increasing load, then, on an ideal strut of square section the maximum permissible fibre stress intensity will be reached, if bending takes place diagonally, with a less deflection than if parallel to the sides, in the ratio of $\frac{1}{\sqrt{2}}$ and in any other direction as $\frac{D}{2\,\delta}$ It therefore follows that the load P required to produce a given maximum stress intensity f in a column of square section is less if deflection takes place diagonally than if parallel to the sides, but as under these conditions the deflection increases very rapidly with an increase of P the difference between the values of P in the two cases is small enough to be negligible. This is clearly shown in Fig. 42, and applies to any isotropic section.

These conditions of equilibrium will only occur in any strut in which the cross section is isotropic, and under perfect alignment and ideal conditions in every way, but such conditions are never attainable. The centre of pressure never exactly coincides with the axis of the strut, but before constructing a strut it is impossible to tell in what direction the departure will lie. There is no reason whatever in building up, say a strut of square section, why the departure should be in a direction parallel to a side, or along a diagonal. The chances that it will lie diagonally are just as great as that it will lie in any other direction, and if it does lie diagonally there is no reason why the extent of the departure should be less than if in any other direction. Everything depends on the direction of the initial departure and its amount, and as it is just as likely to take place in any one direction as in any other, the only safe and reasonable course is to allow for deflection in the worst possible direction.

As v varies with the direction of bending, it follows that for any given initial departure of centre of pressure z, the ratio

$\frac{z}{v}$ varies, and where v takes its least value, $v - z$ is a minimum. Then as v is the greatest permissible amount of deviation of the line of action of P from the axis, and z an initial deviation, the balance $v - z$ is the allowable deflection of the strut under the given load P, and $v - z$ is a minimum where δ is a maximum. Therefore, *for an isotropic section the effect of a given initial deviation of the line of action of* P *is most serious in the direction of the greatest radial dimension*, and not, as might at first appear to be the case, where the radial dimension is least. . . . . (59)

In Fig. 25, let the outer square represent the section of a square column, and the inner square the limiting area for centre of pressure. Then if a circle be drawn with centre at O and radius z, being the amount of an initial deviation of P which may be in any direction, the portion of any radius vector from O intercepted between the circle and the inner square, is the maximum allowable deflection under load. It will be seen that this permissible deflection, in a diagonal direction, equal to $\overline{oe} - z$ is considerably less than that parallel to a side, which $= \overline{oa} - z$

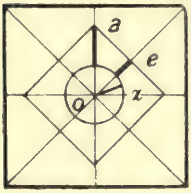


FIG. 25.

Let Fig. 26 represent any section of a bar, and let X X and Y Y be the principal axes. Consider any force P acting in the direction of the arrow and tending to bend the bar. It does not follow that bending will take place in the direction of P. If not, it will deflect in some other direction such as represented by the dotted arrow.

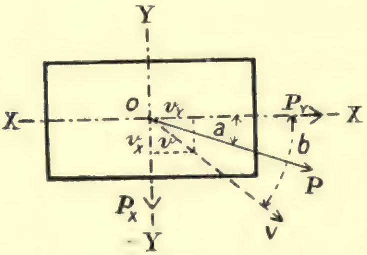


FIG. 26.

Resolve the force P into two components, P_Y acting along

O X and tending to cause bending about Y Y and P_X acting along O Y and tending to cause bending about X X Let v_Y be the deflection resulting from the action of P_Y and v_X from P_X. Then v the diagonal of the rectangle formed by v_Y and v_X is the resulting deflection due to P.

Let a be the angle between O X and O P and b the angle between O X and O V. Then

$$P_Y = P \cdot \cos a \qquad \text{and} \qquad P_X = P \cdot \sin a$$

If the bar is under the conditions of a cantilever or a beam bent by P we have, by equation (1),

$$v = \frac{f \cdot S^2}{E \cdot \delta \times \text{const.}}$$

and as

$$f = \frac{M \cdot \delta}{I_O} = \frac{P \cdot S \cdot \delta \times \text{const.}}{I_O}$$

$$v = \frac{P \cdot S^3}{E \cdot I_O} \times \frac{\text{const.}}{\text{const.}}$$

Now, as S and E remain constant

$$v \quad \text{varies as} \quad \frac{P}{I_O}$$

therefore

$$v_X \quad \text{varies as} \quad \frac{P_X}{I_{OX}} \quad \text{as} \quad \frac{P \cdot \sin a}{I_{OX}}$$

and

$$v_Y \quad \text{varies as} \quad \frac{P_Y}{I_{OY}} \quad \text{as} \quad \frac{P \cdot \cos a}{I_{OY}}$$

Then

$$\frac{v_X}{v_Y} = \frac{I_{OY} \cdot P \cdot \sin a}{I_{OX} \cdot P \cdot \cos a} = \frac{I_{OY}}{I_{OX}} \tan a$$

but

$$\frac{v_X}{v_Y} = \tan b$$

therefore

$$\tan b = \frac{I_{OY}}{I_{OX}} \tan a \quad . \quad . \quad . \quad (60)$$

From the above it will be seen that if the section is isotropic

$$I_{OY} = I_{OX} \quad \text{and} \quad \tan b = \tan a$$

so that in an isotropic section bending will always take place in the direction of action of the deflecting force. . (61)

If the section is not isotropic the direction of O Y will lie to one side, or the other, of O P according as I_{OY} is greater or less than I_{OX} It will always lie on that side of O P towards which I_O is less. . . . . (62)

CHAPTER V.

ELLIPTIC FUNCTIONS.

THERE are many factors and ratios the value of which cannot be expressed exactly in ordinary arithmetical units. For example, in a square having sides of unit length the length of the diagonal is $\sqrt{(\text{length of side})^2 \times 2} = \sqrt{2}$ The diagonal has a definite length just as certainly as has the side, but while the length of the side, in this case, contains a definite number of units of length the length of the diagonal $= \sqrt{2}$, and if we extract the square root of 2 we obtain the value $1 \cdot 414 +$ having an incommensurable fraction. If the process of extracting the root was continued the figures following the decimal point could be added to without limit and at any period still be no nearer the end. Many such cases are well known, such as the ratio of the circumference of a circle to the diameter, and so on.

Mathematical processes often involve the finding of integrals which again involve the summation of series, and most of these series have an infinite number of terms. Many such infinite series have finite sums readily expressed in arithmetical units, but others, like the ratio π, of the circumference of a circle to the diameter, cannot be so expressed. In such cases the value can be calculated to any degree of accuracy required, but not absolutely. In the higher mathematics certain series and summations are repeatedly developed, and those which occur most frequently have been evaluated and tabulated. Series of such a nature occur in connection with problems of the curves of bent struts, and these can be most readily explained and developed from equations to ellipses, hyperbolas, etc., in connection with which they also occur.

In Fig. 27, $A\,B\,A_1\,B_1$ is an ellipse described about the centre O $A\,A_1$ is the major axis $= 2a$ and $B\,B_1$ the minor axis $= 2b$ $A\,Y\,A_1\,Y_1$ is a circle described about the centre O with radius $R = O\,A =$ the semi-axis major. $F\,F_1$ are the two focal points of the ellipse.

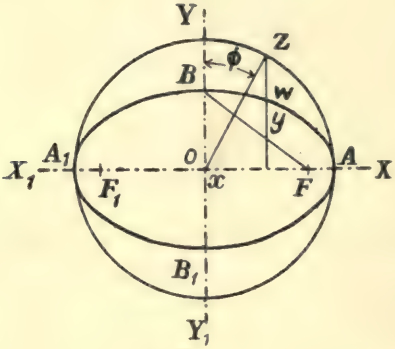


FIG. 27.

Let the ellipse be referred to a pair of rectangular axes $X\,X_1$ and $Y\,Y_1$ coinciding with $A\,A_1$ and $B\,B_1$ respectively, and let distances measured from O along O X be called x and along O Y be called y. Let any line representing a value of y be produced to cut the circle at Z and join O Z. Let the angle Y O Z corresponding to the arc Y Z of the circle be called ϕ.

Now join B F and, from the properties of the ellipse,

$$B\,F = a \qquad \text{Let } O\,F = a \cdot e \qquad \text{then}$$

$$a \cdot e = \sqrt{a^2 - b^2} \qquad \text{and} \quad e = \frac{\sqrt{a^2 - b^2}}{a}$$

or

$$e^2 = \frac{a^2 - b^2}{a^2}$$

The value e is called the eccentricity of the ellipse.

The equation to the ellipse is

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$$

wherefore

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

and

$$y = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}} \quad . \quad . \quad (63)$$

By differentiation, we have

$$\frac{dy}{dx} = -\frac{b}{a} \cdot \frac{x}{(a^2 - x^2)^{\frac{1}{2}}}$$

giving

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{b^2 x^2}{a^2 (a^2 - x^2)} \quad . \quad . \quad (64)$$

Calling s the length of an arc of the ellipse corresponding to x we have

$$s = \int_0^x \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{1}{2}} dx$$

$$= \int_0^x \left\{ 1 + \frac{b^2 x^2}{a^2 (a^2 - x^2)} \right\}^{\frac{1}{2}} dx$$

and, substituting the value of e

$$s = \int_0^x \left\{ \frac{a^2 - e^2 x^2}{a^2 - x^2} \right\}^{\frac{1}{2}} dx$$

therefore

$$s = \int_0^x \left\{ \frac{1 - e^2 \left(\frac{x}{a}\right)^2}{1 - \left(\frac{x}{a}\right)^2} \right\}^{\frac{1}{2}} dx$$

Now, $\sin\phi = \frac{x}{r} = \frac{x}{a}$ therefore $x = a \cdot \sin\phi$ and by differentiation, $dx = a \cdot \cos\phi \cdot d\phi$

Substituting these values, noting that if $x = 0$ then $\phi = 0$

$$s = \int_0^\phi a \left\{ \frac{1 - e^2 \cdot \sin^2\phi}{1 - \sin^2\phi} \right\}^{\frac{1}{2}} \cdot \cos\phi \cdot d\phi$$

which reduces to

$$s = a \int_0^\phi (1 - e^2 \cdot \sin^2\phi)^{\frac{1}{2}} \cdot d\phi \quad . \quad (65)$$

For the quadrant of the ellipse let s become S then

$$S = a \int_0^{\frac{\pi}{2}} (1 - e^2 \cdot \sin^2\phi)^{\frac{1}{2}} \cdot d\phi \quad . \quad (66)$$

The lengths of circular and elliptical arcs, such as Y Z and B W cannot be evaluated from their equations directly, but both cases involve the summation of a series.

In the case of a circle the semi-circumference is $R \cdot \pi$ and π the value of the summation of the corresponding series. The length of any arc of the circle $= R \cdot \phi$ (ϕ being in circular measure), and the length of the arc is therefore the same proportion of $R \cdot \pi$ that ϕ° in degrees is of 180°. Having once calculated the value of π it is only necessary to divide it in the ratio $\frac{\phi^\circ}{180^\circ}$ to obtain the length of the corresponding arc, which then $= \frac{R\,\pi\,\phi^\circ}{180^\circ}$ Values of ϕ are tabulated and are known as the circular measures of angles, being the lengths of arcs, at unit radius, subtending the corresponding angles.

In the case of the ellipse, it is much more difficult to evaluate the length of arcs, as they depend not only on the angle ϕ but also on the eccentricity of the ellipse, there being two variables, and therefore an infinite number of values of either variable for any particular value assigned to the other. Even for any given value of e, the length of the arc does not vary directly as ϕ and it is impossible to tabulate the lengths under all conditions.

Tables of many such arcs have been constructed by Legendre, and the values are known as "Elliptic Functions."

In the processes of evolving these functions the value e is expressed as a function of an angle θ, so that $e = \sin\theta$ wherefore, in the notation of Inverse Functions, $\theta = \sin^{-1} e$

Substituting this value in equations (65) and (66), we have

$$s = a\int_0^{\phi} (1 - e^2 \cdot \sin^2\phi)^{\frac{1}{2}} \cdot d\phi$$

$$= a\int_0^{\phi} (1 - \sin^2\theta \cdot \sin^2\phi)^{\frac{1}{2}} \,.\, d\phi = a \cdot \mathsf{E}\,(\theta, \phi) \quad (67)$$

and

$$S = a\int_0^{\frac{\pi}{2}} (1 - e^2 \cdot \sin^2\phi)^{\frac{1}{2}} \,.\, d\phi$$

$$= a\int_0^{\frac{\pi}{2}} (1 - \sin^2\theta \cdot \sin^2\phi)^{\frac{1}{2}} \cdot d\phi = a \cdot \mathsf{E}_1(\theta) \quad (68)$$

the values of the Integrals being represented by E and E_1 respectively.

In the case of (67), the integral is a function of the two angles θ and ϕ the function being given the symbol E while θ and ϕ are generally expressed in degrees ; for example,

if $\theta = 6°$ and $\phi = 75°$, the equation would read

$$s = a \cdot \mathsf{E}\,(6°, 75°)$$

When ϕ is 90°, the arc is the quadrant of the ellipse. Then, as in equation (68), ϕ is omitted from the final expression and E written as E_1 If, as before, $\theta = 6°$ the length of the quadrant is expressed as $S = a \cdot \mathsf{E}_1\,(6°)$

In these functions the angle θ is called the Modulus, and the angle ϕ the Amplitude. For a given eccentricity θ becomes a constant, and only ϕ varies.

When ϕ is 90° and the arc is the quadrant of the ellipse, the value E_1 is called the Complete Function, and it is to the ellipse what $\frac{\pi}{2}$ is to the circle.

$\theta = 0°$ when $e = 0$ and the ellipse, having no eccentricity, becomes a circle. The quadrant is then $\frac{\pi}{2}$ and therefore

$$E_1\,(0°) = \frac{\pi}{2} \quad . \quad . \qquad (69)$$

At the other extreme limit, when $e = 1$, the ellipse flattens until it coincides with the major axis and becomes a straight line, the minor axis becoming 0, and vanishing. The angle θ is then 90°, and the length of the quadrant being equal to the semi-axis major,

$$E_1\,(90°) = 1$$

Functions derived in a similar manner from the equations to the hyperbola are designated by the symbols F and F_1 These also represent the lengths of arcs, but the relations between the various angles and factors are not so easily illustrated as in the case of the ellipse.

These integrals are expressed thus

$$s = a \cdot \int_0^{\phi} \frac{d\,\phi}{(1 - e^2 \cdot \sin^2\phi)^{\frac{1}{2}}}$$

$$= a \cdot \int_0^{\phi} \frac{d\,\phi}{(1 - \sin^2\theta \cdot \sin^2\phi)^{\frac{1}{2}}} = a \times F\,(\theta\,,\phi) \qquad (70)$$

and

$$S = a \cdot \int_0^{\frac{\pi}{2}} \frac{d\,\phi}{(1 - e^2 \cdot \sin^2\phi)^{\frac{1}{2}}}$$

$$= a \cdot \int_0^{\frac{\pi}{2}} \frac{d\,\phi}{(1 - \sin^2\theta \cdot \sin^2\phi)^{\frac{1}{2}}} = a \times F_1\,(\theta) \qquad (71)$$

Here, as before, F_1 is the complete function, corresponding to $\theta = 90°$, and, again, as $\theta = 0°$ when $e = 0$ we have

$$F_1(0°) = \frac{\pi}{2} \quad . \quad . \quad (72)$$

As θ increases, F_1 also increases until,

when $\theta = \frac{\pi}{2} = 90°$

$$F_1\left(\frac{\pi}{2}\right) \text{ becomes infinite} \quad . \quad . \quad (73)$$

The values F and F_1 are called Elliptic Functions of the First Kind, and E and E_1 are called Elliptic Functions of the Second Kind.

It is good practice to work out some strut problems by the exact formulæ, as well as by the alternative equations given, and for that purpose is appended the following table of values of F_1 and E_1 for a number of different values of θ If other values are required, voluminous tables are to be found in Legendre's works.

As the differences between successive values of F_1 and E_1 are exceedingly small, instead of giving the actual numerical values, the tables give the common logarithms of the values. For the lower and higher values of θ the logarithms are extended to 14 places, and for intermediate values to 12 places. Note that the values of θ are not given in degrees, minutes, etc., but in degrees and decimals.

TABLE OF COMPLETE ELLIPTIC FUNCTIONS.

Modulus Angle θ. Degrees	LOG. F_1	LOG. E_1
0·0	0·196 119 877 030 15	0·196 119 877 030 15
0·1	0·196 120 207 764 42	0·196 119 546 296 02
0·2	0·196 121 199 968 48	0·196 118 554 093 84
0·3	0·196 122 853 646 11	0·196 116 900 424 39
0·4	0·196 125 168 803 61	0·196 114 585 288 93
0·5	0·196 128 145 449 79	0·196 111 608 689 19
0·6	0·196 131 783 595 98	0·196 107 970 627 54
0·7	0·196 136 083 256 04	0·196 103 671 106 64
0·8	0·196 141 044 446 35	0·196 098 710 129 83
0·9	0·196 146 667 185 80	0·196 093 087 700 84
1·0	0·196 152 951 495 81	0·196 086 803 823 97
1·1	0·196 159 897 400 35	0·196 079 858 503 87
1·2	0·196 167 504 925 74	0·196 072 251 746 30
1·3	0·196 175 774 101 10	0·196 063 983 556 62
1·4	0·196 184 704 957 89	0·196 055 053 941 25
1·5	0·196 194 297 530 11	0·196 045 462 907 07
1·6	0·196 204 551 854 35	0·196 035 210 461 31
1·7	0·196 215 467 969 65	0·196 024 296 611 89
1·8	0·196 227 045 917 60	0·196 012 721 367 12
1·9	0·196 239 285 742 35	0·196 000 484 735 83
2·0	0·196 252 187 490 54	0·195 987 586 727 42
2·1	0·196 265 751 211 33	0·195 974 027 351 73
2·2	0·196 279 976 956 47	0·195 959 806 619 19
2·3	0·196 294 864 780 17	0·195 944 924 540 65
2·4	0·196 310 414 739 20	0·195 929 381 127 52
2·5	0·196 326 626 892 86	0·195 913 176 391 74
2·6	0·196 343 501 303 00	0·195 896 310 345 75
2·7	0·196 361 038 033 99	0·195 878 783 002 50
2·8	0·196 379 237 152 72	0·195 860 594 375 47
2·9	0·196 398 098 728 66	0·195 841 744 478 65
3·0	0·196 417 622 833 76	0·195 822 233 326 59
3·1	0·196 437 809 542 58	0·195 802 060 934 20
3·2	0·196 458 658 932 16	0·195 781 227 317 18
3·3	0·196 480 171 082 15	0·195 759 732 491 48
3·4	0·196 502 346 074 66	0·195 737 576 473 79
3·5	0·196 525 183 994 43	0·195 714 759 281 23
3·6	0·196 548 684 928 69	0·195 691 280 931 40
3·7	0·196 572 848 967 25	0·195 667 141 442 50
3·8	0·196 597 676 202 46	0·195 642 340 833 30
3·9	0·196 623 166 729 25	0·195 616 879 122 95

TABLE OF COMPLETE ELLIPTIC FUNCTIONS—*continued.*

Modulus Angle θ. Degrees	LOG. F_1	LOG. E_1
4·0	0·196 649 320 645 08	0·195 590 756 331 28
4·1	0·196 676 138 049 96	0·195 563 792 478 62
4·2	0·196 703 619 046 50	0·195 536 527 585 73
4·3	0·196 731 763 739 84	0·195 508 421 674 06
4·4	0·196 760 572 237 68	0·195 479 654 765 50
4·5	0·196 790 044 650 35	0·195 450 226 882 48
4·6	0·196 820 181 090 64	0·195 420 138 048 15
4·7	0·196 850 981 674 00	0·195 389 388 285 96
4·8	0·196 882 446 518 46	0·195 357 977 619 94
4·9	0·196 914 575 744 55	0·195 325 906 074 82
5·0	0·196 947 369 475 26	0·195 293 173 675 78
5·1	0·196 980 827 836 82	0·195 259 780 448 55
5·2	0·197 014 950 957 06	0·195 225 726 419 56
5·3	0·197 049 738 967 06	0·195 191 011 615 52
5·4	0·197 085 192 000 40	0·195 155 636 063 67
5·5	0·197 121 310 193 01	0·195 119 599 792 43
5·6	0·197 158 093 683 74	0·195 082 902 829 80
5·7	0·197 195 542 613 73	0·195 045 545 205 38
5·8	0·197 233 657 126 96	0·195 007 526 948 53
5·9	0·197 272 437 369 88	0·194 968 848 098 62
6·0	0·197 311 883 491 58	0·194 929 508 659 37
7·0	0·197 743 002 715 22	0.194 499 790 128 70
8·0	0·198 240 893 316 89	0·194 004 052 701 49
9·0	0·198 805 747 967 31	0·193 442 337 033 72
10·0	0·199 437 786 246 87	0·192 814 690 130 52
11·0	0·200 137 255 115 91	0·192 121 165 702 08
12·0	0·200 904 429 447 80	0·191 361 824 566 65
13·0	0.201 739 612 628 24	0·190 536 735 099 87
14·0	0·202 643 137 221 85	0·189 645 973 738 18
15·0	0·203 615 365 712 62	0·188 689 625 529 78
16·0	0·204 656 691 320	0·187 667 784 743
17·0	0·205 767 538 891	0·186 580 555 528
18·0	0·206 948 365 890	0·185 428 052 646
19·0	0·208 199 663 461	0·184 210 402 247
20·0	0·209 521 957 602	0·182 927 742 730
21·0	0·210 915 810 437	0·181 580 225 657
22·0	0·212 383 821 593	0·180 168 016 750
23·0	0·213 920 629 710	0·178 691 296 961
24·0	0·215 532 914 065	0·177 150 263 626
25·0	0·217 219 396 341	0·175 545 131 698

TABLE OF COMPLETE ELLIPTIC FUNCTIONS—*continued.*

Modulus Angle θ. Degrees	Log. F_1	Log. E_1
26·0	0·218 980 842 547	0·173 876 135 083
27·0	0·220 818 065 097	0·172 143 528 063
28·0	0·222 731 925 065	0·170 347 586 827
29·0	0·224 723 334 635	0·168 488 611 109
30·0	0·226 793 259 758	0·166 566 925 942
31·0	0·228 942 723 038	0·164 582 883 541
32·0	0·231 172 806 867	0·162 536 865 306
33·0	0·233 484 656 838	0·160 429 283 980
34·0	0·235 879 485 458	0·158 260 585 943
35·0	0·238 358 576 191	0·156 031 253 676
36·0	0·240 923 287 876	0·153 741 808 383
37·0	0·243 575 059 539	0·151 392 812 807
38·0	0·246 315 415 669	0·148 984 874 232
39·0	0·249 145 971 982	0·146 518 647 689
40·0	0·252 068 441 749	0·143 994 839 391
41·0	0·255 084 642 742	0·141 414 210 399
42·0	0·258 196 504 876	0·138 777 580 537
43·0	0·261 406 078 624	0·136 085 832 576
44·0	0·264 715 544 316	0·133 339 916 726
45·0	0·268 127 222 412	0·130 540 855 400
46·0	0·271 643 584 898	0·127 689 748 344
47·0	0·275 267 267 932	0·124 787 778 097
48·0	0·279 001 085 928	0·121 836 215 848
49·0	0·282 848 047 272	0·118 836 427 682
50·0	0·286 811 371 888	0·115 789 881 279
51·0	0·290 894 510 955	0·112 698 153 070
52·0	0·295 101 169 067	0·109 562 935 908
53·0	0·299 435 329 234	0·106 386 047 271
54·0	0·303 901 281 165	0·103 169 438 054
55·0	0·308 503 653 367	0·099 915 201 993
56·0	0·313 247 449 710	0·096 625 585 767
57·0	0·318 138 091 215	0·093 302 999 831
58·0	0·323 181 464 010	0·089 950 030 061
59·0	0·328 383 974 580	0·086 569 450 247
60·0	0·333 752 613 698	0·083 164 235 547
61·0	0·339 295 030 747	0·079 737 576 960
62·0	0·345 019 620 541	0·076 292 896 937
63·0	0·350 935 625 280	0·072 833 866 233
64·0	0·357 053 254 962	0·069 364 422 135
65·0	0·363 383 830 425	0·065 888 788 224

TABLE OF COMPLETE ELLIPTIC FUNCTIONS—
continued.

Modulus Angle θ. Degrees	LOG. F_1	LOG. E_1
66·0	0·369 939 954 408	0·062 411 495 834
67·0	0·376 735 717 520	0·058 937 407 456
68·0	0·383 786 948 147	0 055 471 742 309
69·0	0·391 111 518 109	0·052 020 104 418
70·0	0·398 729 719 921	0·048 588 513 576
71·0	0·406 664 736 899	0·045 183 439 678
72·0	0·414 943 235 296	0·041 811 841 030
73·0	0·423 596 118 937	0·038 481 207 422
74·0	0·432 659 503 542	0·035 199 608 970
75·0	0·442 175 993 072	0·031 975 752 057
76·0	0·452 196 379 091 74	0·028 819 044 169 78
77·0	0·462 781 945 226 85	0·025 739 670 047 81
78·0	0·474 007 658 076 26	0·022 748 682 517 70
79·0	0·485 966 692 724 94	0·019 858 112 804 32
80·0	0·498 777 032 133 31	0·017 081 107 271 63
81·0	0·512 591 410 716 56	0·014 432 101 009 44
82·0	0·527 612 893 033 96	0·011 927 044 377 36
83·0	0·544 120 466 631 80	0·009 583 708 479 23
84·0	0·562 513 640 468 45	0·007 422 113 537 40
85·0	0·583 396 259 318 33	0·005 465 159 414 92
86·0	0·607 750 689 909 07	0·003 739 616 163 78
87·0	0·637 355 014 889 22	0·002 277 791 659 68
88·0	0·676 027 253 174 92	0·001 120 806 578 23
89·0	0·735 192 343 119 46	0·000 326 283 450 30
89·9	0·888 578 886 838 43	0·000 004 787 090 76
90·0	Infinite	Zero

CHAPTER VI.

ELASTIC CURVES OF BENDING.

WHEN a bar is bent into a curved form the fibres on the convex side are lengthened and those on the concave side are shortened.

Let A B in Fig. 28 represent a bar of uniform cross section, and originally straight, bent by some external forces to a curve of uniform radius. As the outer fibres are lengthened and the inner ones shortened, there must be some line of fibres between these extremes which, although curved, has not been altered in length. Let X X be that line. It is usually called the neutral axis of the bar, although it is, in reality, a neutral plane. It may, however, without leading to any confusion be called the neutral axis. If the modulus of elasticity of the material is the same for tensile and compressive stresses, the neutral axis will pass through the centre of gravity of the cross section.

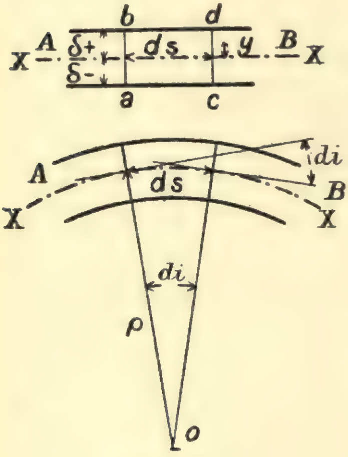


FIG. 28.

The alteration in length, or the strain in any fibre, is proportional to its distance y from the neutral axis.

Let δ_+ be the distance of the extreme fibre from the axis on the convex side, and δ_- the corresponding distance on the concave side. Let y be the distance of any other fibre from the axis, and let ρ be the radius of curvature of the axis. The reciprocal $\frac{1}{\rho}$ of the radius is the "Curvature."

Any two planes, such as *a b* and *c d* originally parallel and at right angles to the axis will, after the bar is bent, radiate from the centre O. In Fig. 29 let the distance, along the axis, between two such planes $= \Delta s$, and the angle at O (subtended by Δs) $= \Delta i$ which is also the angle between the tangents to the curved axis at the points where it intersects *a b* and *c d*. Let *m n* be a line, parallel to *c d* drawn through the intersection of *a b* with the axis. Then *a m* is approximately the amount by which the fibre *a c* has been lengthened, and *b n* the amount by which *b d* has been shortened.

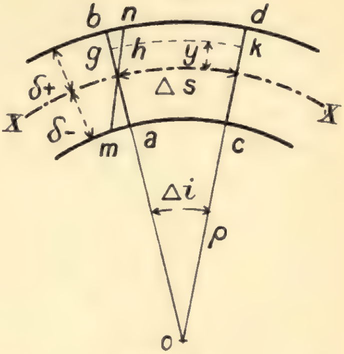


FIG. 29.

Then, considering the short arcs to be nearly straight lines, we have,

$$\frac{\overline{gh}}{y} = \frac{\overline{bn}}{\delta_+} = \frac{\overline{am}}{\delta_-} = \frac{\Delta s}{\rho}$$

therefore

$$\frac{1}{\rho} = \frac{\overline{gh}}{y \cdot \Delta s} = \frac{\overline{bn}}{\delta_+ \cdot \Delta s} = \frac{\overline{am}}{\delta_- \cdot \Delta s} \quad . \quad (74)$$

and

$$\Delta s = \overline{gh}\,\frac{\rho}{y} = \overline{bn}\,\frac{\rho}{\delta_+} = \overline{am}\,\frac{\rho}{\delta_-} \quad . \quad (75)$$

But $\overline{gh}$, $\overline{bn}$ and $\overline{am}$ are the alterations in length of the fibres at distances, y , δ_+ and δ_- respectively, from the axis and the strains in these fibres are

$$\frac{\overline{gh}}{\overline{hk}} , \quad \frac{\overline{bn}}{\overline{nd}} \text{ and } \frac{\overline{am}}{\overline{mc}}$$

or their equivalents

$$\frac{\overline{gh}}{\Delta s}, \frac{\overline{bn}}{\Delta s} \text{ and } \frac{\overline{am}}{\Delta s}$$

If the arc Δs is continually reduced, the error caused by the assumption is continually reduced, and the lengths of the chord and the arc become more nearly equal until, as they approach the limit, zero, they eventually coincide and the ratios in the equations become exact.

Equation (74) then becomes

$$\frac{1}{\rho} = \frac{\overline{gh}}{y \cdot ds} = \frac{\overline{bn}}{\delta_+ \cdot ds} = \frac{\overline{am}}{\delta_- \cdot ds} \quad . \quad (76)$$

Again, if p'' be the stress intensity on any fibre, the corresponding strain is $\frac{p''}{E}$ and therefore, substituting in (76), we have

$$\frac{1}{\rho} = \frac{p''}{Ey} \quad . \quad . \quad . \quad (77)$$

If M is the moment of the external forces, causing the bending moment about the axis X X
then

$$p'' = \frac{My}{I_0}$$

and, generally,

$$\frac{1}{\rho} = \frac{M}{EI_0} \quad . \quad . \quad . \quad (78)$$

In Fig. 29, the angle Δi if expressed in circular measure gives, approximately, by similar triangles,

$$\frac{\Delta i}{1} = \frac{\Delta s}{\rho}$$

and when Δs becomes ds, Δi becomes di, and the ratio becomes absolute, and therefore

$$di = \frac{ds}{\rho} \quad \text{or} \quad ds = \rho \cdot di \qquad (79)$$

If a flexible band in the form of a circular ring is acted upon by a uniform internal pressure, such as a hydraulic pressure, it will be subjected to a tensile stress, uniform in amount at all sections of its circumference. The tension must be uniform, because each point in the ring is similarly situated with regard to all the other points and in relation to the pressure. In Fig. 30, let the ring be of radius ρ and the intensity of the internal pressure p. Consider half of the ring, on one side of a diameter $a\,b$ The forces acting on, say, the left hand side of the plane $a\,b$ are the two tangential forces T T which are the tensions in the ring at the points a and b; and the pressure of intensity p acting over the diameter $a\,b$ the total amount being $2\rho \times p$. These forces are in equilibrium, and therefore

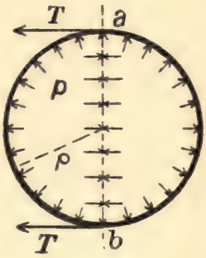


FIG. 30.

$$2\,T = 2\rho p \quad \text{or} \quad T = \rho p$$

This ratio is true for any point in the ring, and is irrespective of the actual dimensions or the amount of the intensity of the pressure. If, therefore, a curved band is of varying radius, and acted upon by a varying pressure; and if under these conditions it is in a state of equilibrium, the above ratios must apply and we therefore have, at any point

$$T = \rho p \; . \quad . \quad . \qquad (80)$$

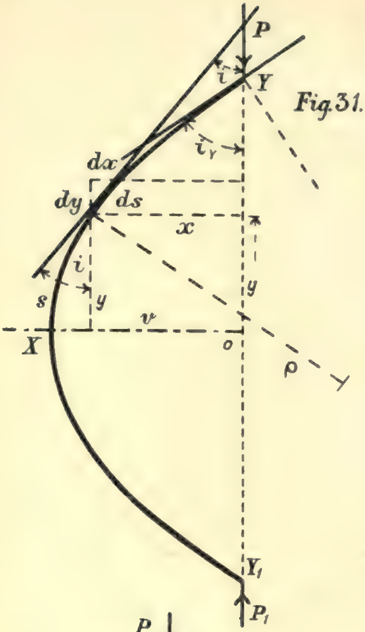


Fig.31.

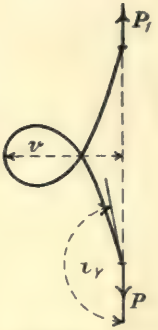


Fig.35.

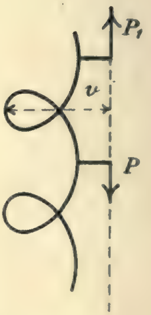


Fig.36.

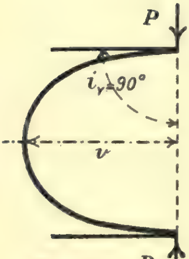


Fig.32.

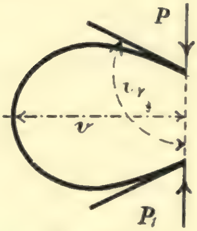


Fig.33.

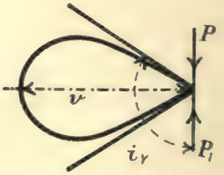


Fig.34.

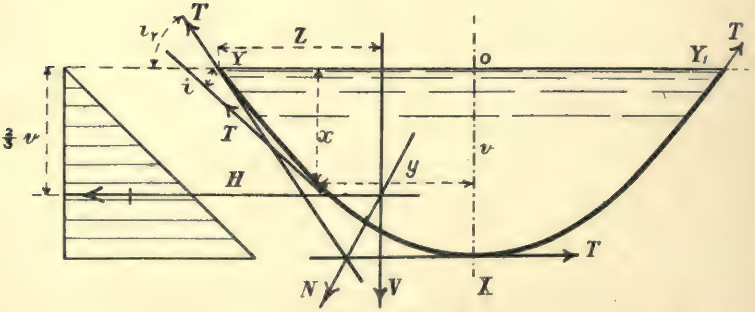


Fig. 37.

If such a band is subjected to a varying pressure, it will at all times assume such a form that the tension and curvature will be in accordance with these equations.

If the pressure is on the convex side of the curve, the stresses will be compressive, but the equations are the same.

In Fig. 31, let Y X Y_1 represent the curve of a spring of uniform cross section throughout, and originally perfectly straight, bent under the action of a pair of equal and opposite forces, P and P_1 acting along a line joining the two ends of the longitudinal axis of the spring.

Through the point O at the centre of the chord Y Y_1 draw a line, O X at right angles to Y Y_1 and intersecting the curve in its centre point X

Referring the curve to O Y and O X as a pair of rectangular axes, let

y = any distance measured along O Y
$-y$ = „ „ O Y_1
x = „ „ O X
ρ = the radius of curvature at any point in the curve
i = the angle made by any tangent to the curve with the axis of Y
s = any distance, measured from X along the curve towards Y
$-s$ = any distance, measured from X along the curve towards Y_1

At the centre X of the curve let x become v

then $i = i_0 = 0$ and $\rho = \rho_v$

For the half curve, X Y let $s = S$ and $y = Y$

then $i = i_Y$.

Let E = the modulus of elasticity of the material of the spring, and let this be equal in both tension and compression. Let I_0 be the moment of inertia of the cross section of the spring about the neutral axis corresponding to the direction of the bending.

The only external forces acting on the spring are P and P_1 and as the whole is in equilibrium, these are resisted by the tendency of the spring to resume its original straight form, and the moments of the internal stresses on any section must equal the bending moment, about its axis, caused by the external forces.

At any point in the axis of the spring there is a bending moment

$$M = P x \quad . \quad . \quad . \quad (81)$$

From (78)

$$\frac{1}{\rho} = \frac{M}{E I_0} = \frac{P}{E I_0} x$$

therefore

$$\rho x = \frac{E I_0}{P} = \text{a constant,}$$

and therefore

$$\rho x = \rho_v v \quad . \quad . \quad . \quad (82)$$

Let the constant

$$\frac{E I_0}{P} = K^2 \quad . \quad . \quad . \quad (83)$$

then

$$\rho x = \rho_v v = K^2 \quad . \quad . \quad (84)$$

and

$$\frac{1}{\rho} = \frac{x}{K^2} \quad . \quad . \quad . \quad (85)$$

Also

$$\frac{1}{\rho_v} = \frac{v}{K^2} \quad . \quad . \quad . \quad (86)$$

Now

$$\sin i = \frac{-d x}{d s}$$

therefore

$$d s = \frac{-d x}{\sin i} \quad . \quad . \quad . \quad (87)$$

the sign of $d\,x$ being minus because x decreases as s or y increases.

Again, from (79), (85), and (87)

$$d\,\imath \quad = \quad \frac{1}{\rho}\,d\,s \quad = \quad \frac{1}{K^2}\cdot x\cdot d\,s \quad = \quad -\frac{1}{K^2}\cdot x\cdot\frac{d\,x}{\sin i}$$

therefore

$$-\sin i\cdot d\,i \quad = \quad \frac{1}{K^2}\cdot x\cdot d\,x$$

and, integrating,

$$\cos i \quad = \quad \frac{1}{K^2}\int x\cdot d\,x \quad = \quad \frac{x^2}{2\,K^2}+c$$

but, when $x \quad = \quad v$

$$i \quad = \quad i_0 \quad = \quad 0^\circ \quad \text{and} \quad \cos i_0 \quad = \quad 1$$

wherefore

$$\cos i_0 \quad = \quad \frac{v^2}{2\,K^2}+c \quad = \quad 1$$

and

$$c \quad = 1-\frac{v^2}{2\,K^2}$$

therefore

$$\cos i \ = \ 1-\frac{v^2}{2\,K^2}+\frac{x^2}{2\,K^2} \ = \ 1-\frac{v^2}{2\,K^2}\left(1-\frac{x^2}{v^2}\right) \qquad (88)$$

By (84) we had $\rho_v\,v \quad = \quad K^2$ and, substituting in (88), we have also

$$\cos i \quad = \quad 1-\frac{v}{2\,\rho_v}\left(1-\frac{x^2}{v^2}\right) \quad . \qquad . \quad (89)$$

and, as x becomes 0 at Y

$$\cos i_Y \quad = \quad 1-\frac{v^2}{2\,K^2} \quad = \quad 1-\frac{v}{2\,\rho_v} \quad . \quad (90)$$

By (88), $$\cos i = 1 - \frac{v^2}{2\,K^2} + \frac{x^2}{2\,K^2}$$

from which

$$\begin{aligned} x^2 &= v^2 - 2\,K^2 + 2\,K^2 \cdot \cos i \\ &= v^2 - 2\,K^2\,(1 - \cos i) \end{aligned}$$

and

$$x = \left\{v^2 - 2\,K^2\,(1 - \cos i)\right\}^{\frac{1}{2}} \quad . \quad . \quad (91)$$

or, by substitution,

$$x = \left\{v^2 - 2\,\rho_v\,v\,(1 - \cos i)\right\}^{\frac{1}{2}} \quad . \quad . \quad (92)$$

TO FIND THE LENGTH OF THE CURVE.

By (87) $$d\,s = \frac{-\,d\,x}{\sin i}$$

from which, by integration,

$$s = \int_v^x \frac{-\,d\,x}{\sin i} = \int_v^x \frac{-\,d\,x}{(1 - \cos^2 i)^{\frac{1}{2}}}$$

and, substituting from (88)

$$s = \int_v^x \frac{-\,d\,x}{\left[1 - \left\{1 - \frac{v^2}{2\,K^2}\left(1 - \frac{x^2}{v^2}\right)\right\}^2\right]^{\frac{1}{2}}}$$

then, by reduction,

$$s = \int_v^x \frac{-\,d\,x}{\left\{\frac{v^2}{K^2}\left(1 - \frac{x^2}{v^2}\right) - \frac{v^4}{4\,K^4}\left(1 - \frac{x^2}{v^2}\right)^2\right\}^{\frac{1}{2}}}$$

$$= \frac{K}{v}\int_v^x \frac{-\,d\,x}{\left\{\left(1 - \frac{x^2}{v^2}\right) - \frac{v^2}{4\,K^2}\left(1 - \frac{x^2}{v^2}\right)^2\right\}^{\frac{1}{2}}} \quad . \quad (93)$$

Now, let ϕ be the angle whose cosine $= \frac{x}{v}$

then

$$x = v \cdot \cos\phi$$

$$dx = -v \cdot \sin\phi \cdot d\phi$$

and

$$\sin^2\phi = 1 - \cos^2\phi = 1 - \frac{x^2}{v^2}$$

Substituting these values in (93), and noting that

$$\cos\phi = 1 \quad \text{when} \quad x = v$$

and therefore $\phi = 0^\circ$

we have

$$s = \frac{K}{v} \cdot \int_0^{\phi} \frac{v \cdot \sin\phi \cdot d\phi}{\left\{\sin^2\phi - \frac{v^2}{4K^2} \cdot \sin^4\phi\right\}^{\frac{1}{2}}}$$

and

$$s = K \cdot \int_0^{\phi} \frac{d\phi}{\left\{1 - \left(\frac{v}{2K}\right)^2 \cdot \sin^2\phi\right\}^{\frac{1}{2}}} \qquad (94)$$

or

$$s = \sqrt{\rho_v v} \cdot \int_0^{\phi} \frac{d\phi}{\left\{1 - \left(\frac{\sqrt{v}}{2\sqrt{\rho_v}}\right)^2 \cdot \sin^2\phi\right\}^{\frac{1}{2}}} \qquad (95)$$

From (70), it will be seen that these integrals depend upon Elliptic Functions, and putting

θ for the angle whose sine $= \frac{v}{2K} = \frac{\sqrt{v}}{2\sqrt{\rho_v}}$. (96)

(94) becomes

$$s = \mathrm{K}\int_0^{\phi} \frac{d\phi}{(1 - \sin^2\theta \cdot \sin^2\phi)^{\frac{1}{2}}} = \mathrm{K}\left\{\mathsf{F}(\theta, \phi) - \mathsf{F}(\theta, 0) + c\right\}$$

or

$$s = \sqrt{\rho_v v} \cdot \int_0^{\phi} \frac{d\phi}{(1 - \sin^2 \cdot \sin^2\phi)^{\frac{1}{2}}} = \sqrt{\rho_v v}\left\{\mathsf{F}(\theta,\phi) - \mathsf{F}(\theta,0) + c\right\}$$

F being the symbol for an Elliptic Function of the First Kind.

Now $s = 0$ when $x = v$ therefore $c = 0$ and the general equations are

$$s = \mathrm{K} \int_0^{\phi} \frac{d\phi}{(1 - \sin^2\theta \cdot \sin^2\phi)^{\frac{1}{2}}}$$

$$= \mathrm{K} \times \mathsf{F}\left\{\sin^{-1}\left(\frac{v}{2\,\mathrm{K}}\right), \phi\right\} \quad . \quad . \quad (97)$$

or

$$s = \sqrt{\rho_v v} \cdot \int_0^{\phi} \frac{d\phi}{(1 - \sin^2\theta \cdot \sin^2\phi)^{\frac{1}{2}}}$$

$$= \sqrt{\rho_v v} \times \mathsf{F}\left\{\sin^{-1}\left(\frac{\sqrt{v}}{2\sqrt{\rho_v}}\right), \phi\right\} \quad . \quad (98)$$

and

$$\mathrm{S} = \mathrm{K} \cdot \int_0^{\frac{\pi}{2}} \frac{d\phi}{(1 - \sin^2\theta \cdot \sin^2\phi)^{\frac{1}{2}}}$$

$$= \mathrm{K} \times \mathsf{F}_1\left\{\sin^{-1}\left(\frac{v}{2\,\mathrm{K}}\right)\right\} \quad . \quad . \quad (99)$$

or

$$\mathrm{S} = \sqrt{\rho_v v} \cdot \int_0^{\frac{\pi}{2}} \frac{d\phi}{(1 - \sin^2\theta \cdot \sin^2\phi)^{\frac{1}{2}}}$$

$$= \sqrt{\rho_v v} \times \mathsf{F}_1\left\{\sin^{-1}\left(\frac{\sqrt{v}}{2\sqrt{\rho_v}}\right)\right\} \quad . \quad (100)$$

In these equations the angle θ is finally expressed in the notation of inverse functions. Therefore the expression

$$\sin^{-1}\left(\frac{v}{2\,\mathrm{K}}\right) \text{ means the same as } \theta$$

because $\sin\theta = \dfrac{v}{2\,\mathrm{K}}$ therefore $\theta = \sin^{-1}\left(\dfrac{v}{2\,\mathrm{K}}\right)$ The expression $\sin^{-1}\left(\dfrac{v}{2\,\mathrm{K}}\right)$ means "the angle θ whose sine is $\dfrac{v}{2\,\mathrm{K}}$" and so on.

By (90), we have

$$\cos i_{\mathrm{Y}} = 1 - \frac{v}{2\,\rho_v}$$

and by (96),

$$\sin\theta = \frac{\sqrt{v}}{2\sqrt{\rho_v}}$$

$$= \frac{\left(\dfrac{v}{\rho_v}\right)^{\frac{1}{2}}}{2} = \frac{\left\{2 - \left(2 - \dfrac{v}{\rho_v}\right)\right\}^{\frac{1}{2}}}{2}$$

which, by substitution, can be transformed into

$$\sin\theta = \frac{(2 - 2\cos i_{\mathrm{Y}})^{\frac{1}{2}}}{2}$$

From this equation of trigonometrical ratios, it follows that

$$\theta = \frac{i_{\mathrm{Y}}}{2} \quad . \quad . \quad . \quad (101)$$

That is, the modulus angle θ is half the angle i_{Y} which the tangent to the curve, at its end, makes with the axis of Y *or* half the angle at which the curve meets the axis of Y.

TO FIND LENGTHS MEASURED ALONG THE CHORD.

We have

$$dy = \cos i \cdot ds = \frac{-\cos i}{\sin i}\,dx = \frac{-\cos i}{(1 - \cos^2 i)^{\frac{1}{2}}}\,dx$$

and from (88),

$$\cos i = 1 - \frac{v^2}{2\,K^2}\left(1 - \frac{x^2}{v^2}\right)$$

Substituting in this the equivalents in terms of ϕ we have

$$\cos i = 1 - \frac{v^2}{2\,K^2} \cdot \sin^2 \phi$$

then

$$dy = v \cdot \frac{1 - \dfrac{v^2}{2\,K^2} \cdot \sin^2 \phi}{\left\{\dfrac{v^2}{K^2} - \left(\dfrac{v^2}{2\,K^2}\right)^2 \cdot \sin^2 \phi\right\}^{\frac{1}{2}}}\,d\phi$$

$$= K \cdot \frac{2\left\{1 - \left(\dfrac{v}{2\,K}\right)^2 \cdot \sin^2 \phi\right\} - 1}{\left\{1 - \left(\dfrac{v}{2\,K}\right)^2 \cdot \sin^2 \phi\right\}^{\frac{1}{2}}}\,d\phi$$

which reduces to

$$dy = K\left[2\left\{1 - \left(\frac{v}{2\,K}\right)^2 \cdot \sin^2 \phi\right\}^{\frac{1}{2}} d\phi \right.$$

$$\left. - \frac{d\phi}{\left\{1 - \left(\dfrac{v}{2\,K}\right)^2 \cdot \sin^2 \phi\right\}^{\frac{1}{2}}}\right] \qquad (102)$$

then, by integration

$$y = \mathrm{K}\left[2\int_0^{\phi}\left\{1 - \left(\frac{v}{2\,\mathrm{K}}\right)^2 \cdot \sin^2\phi\right\}^{\frac{1}{2}} d\,\phi - \int_0^{\phi}\frac{d\,\phi}{\left\{1 - \left(\frac{v}{2\,\mathrm{K}}\right)^2 \cdot \sin^2\phi\right\}^{\frac{1}{2}}}\right] \qquad (103)$$

or

$$y = \mathrm{K}\left[2\,\mathsf{E}\left\{\sin^{-1}\left(\frac{v}{2\,\mathrm{K}}\right), \phi\right\} - \mathsf{F}\left\{\sin^{-1}\left(\frac{v}{2\,\mathrm{K}}\right), \phi\right\}\right] \qquad (104)$$

E being an Elliptic Function of the Second Kind, and F an Elliptic Function of the First Kind.

Alternatively, on substituting values of ρ_v in (104) it becomes

$$y = \sqrt{\rho_v\, v}\left[2\,\mathsf{E}\left\{\sin^{-1}\left(\frac{\sqrt{v}}{2\sqrt{\rho_v}}\right), \phi\right\} - \mathsf{F}\left\{\sin^{-1}\left(\frac{\sqrt{v}}{2\sqrt{\rho_v}}\right), \phi\right\}\right] \qquad (105)$$

And for the length of the Half Chord, O Y or O Y_1

$$\mathrm{Y} = \mathrm{K}\left[2\,\mathsf{E}_1\left\{\sin^{-1}\left(\frac{v}{2\,\mathrm{K}}\right)\right\} - \mathsf{F}_1\left\{\sin^{-1}\left(\frac{v}{2\,\mathrm{K}}\right)\right\}\right] \qquad (106)$$

or

$$\mathrm{Y} = \sqrt{\rho_v\, v}\left[2\,\mathsf{E}_1\left\{\sin^{-1}\left(\frac{\sqrt{v}}{2\sqrt{\rho_v}}\right)\right\} - \mathsf{F}_1\left\{\sin^{-1}\left(\frac{\sqrt{v}}{2\sqrt{\rho_v}}\right)\right\}\right] \qquad (107)$$

In all cases the evaluation of s or y involves the extraction of a square root, and the values will, therefore always have the alternative signs $\pm$, being $+$ or $-$ according as x is taken towards Y or Y_1

SINGULAR VALUES OF THE EQUATIONS.

Under certain degrees of bending some of the factors in these equations cancel out, and the equations take much simpler forms. A few of these singular values occur as follows :—

When the deflection v is nil.

In this case, substituting the value $v = 0$ in (100), the algebraic factor, $\sqrt{\rho_v v}$ becomes $\sqrt{\infty \times 0}$ and the value of S is indeterminate, but taking (99)

$$S = K \cdot \int_0^{\frac{\pi}{2}} \frac{d\phi}{(1 - \sin^2\theta \cdot \sin^2\phi)^{\frac{1}{2}}}$$

$$\sin\theta = \frac{v}{2\,K} \text{ also becomes } = 0 \text{ when } v = 0$$

therefore

$$S = K \cdot \int_0^{\frac{\pi}{2}} d\phi = K\frac{\pi}{2} \qquad (108)$$

and, substituting the value of K from (83), we have

$$S = \sqrt{\frac{E\,I_0}{P}} \cdot \frac{\pi}{2} \qquad . \qquad (109)$$

Or, again, by (99),

$$S = K \times F_1\left\{\sin^{-1}\left(\frac{v}{2\,K}\right)\right\}$$

which, when $v = 0$ becomes

$$S = K \times F_1(\sin^{-1} 0^\circ) = K \times F_1(0^\circ)$$

Here, by (72) we find that $F_1(0^\circ) = \frac{\pi}{2}$

so that, as before,

$$S = \sqrt{\frac{E\,I_0}{P}} \cdot \frac{\pi}{2} \quad . \quad (110)$$

This result is of the greatest importance. The meaning of (110) is :—*For any spring of uniform cross section throughout, and with axis straight, and of a given length, having forces* P *acting at its ends along the axis so as to compress it ; there is a definite minimum value of* P *below which it will not cause any bending, or deflection, in the spring.*

This value is therefore

$$P = \frac{E\,I_0\,\pi^2}{4\,S^2} \quad . \quad . \quad (111)$$

and is the expression known as Euler's formula. This equation is exact, and in the foregoing process of arriving at it no substitutions or assumptions of any kind have been used.

When the Tangents to the Curve, at its two ends, become parallel.

Referring to Fig. 32, the angle i_Y is now $90^\circ = \frac{\pi}{2}$ and

$$\cos\frac{\pi}{2} = 0$$

Then, from (90),

$$\cos i_Y = 1 - \frac{v^2}{2\,K^2} = 1 - \left(\frac{v}{\sqrt{2}\,K}\right)^2 = 0$$

therefore

$$\frac{v}{\sqrt{2}\,K} = 1 \quad \text{and} \quad v = \sqrt{2} \cdot K \qquad (112)$$

Or, from (90),

$$\cos i_Y = 1 - \frac{v}{2\rho_v} = 0 \quad \text{therefore} \quad v = 2\rho_v$$

and

$$\rho_v = \frac{v}{2} \quad . \quad . \quad (113)$$

Substituting in (93),

$$s = v^2 \int_v^x \frac{-dx}{(v^4 - x^4)^{\frac{1}{2}}} \quad . \quad (114)$$

and, substituting in (100),

$$S = \frac{1}{\sqrt{2}} \cdot v \cdot F_1 \left\{ \sin^{-1} \sqrt{\frac{1}{2}} \right\} = \frac{v}{\sqrt{2}} \cdot F_1 \left(\frac{\pi}{4} \right) \quad (115)$$

which, reduced to its numerical value, gives

$$S = \pm v \times 1{\cdot}311$$

Again, substituting in (107),

$$Y = \frac{v}{\sqrt{2}} \left\{ 2E_1\left(\frac{\pi}{4}\right) - F_1\left(\frac{\pi}{4}\right) \right\} \quad (116)$$

which, in turn, reduced to its numerical value, becomes

$$Y = \pm v \times 0{\cdot}599$$

When the Deflection v is a Maximum. (Fig. 33.)

If $\rho_v = \frac{v}{2{\cdot}828}$ the ratio $\frac{v}{S}$ is a maximum. That is, for a given length of curve, the deflection v is a maximum.

Then

$$S = \pm v \times 1{\cdot}240 \text{ (about)}$$
$$Y = \pm v \times 0{\cdot}50 \text{ (about)}$$
$$i_Y = 114^\circ{\cdot}28' \text{ (about)}$$

When the two ends of the spring meet. (*Fig.* 34.)

Then

$$\begin{aligned} \rho_v &= \frac{v}{3 \cdot 3 +} \\ S &= \pm\, v \times 1 \cdot 277 \\ \overline{Y} &= \pm\, 0 \\ \overline{i_Y} &= 130^\circ \cdot 42' + \end{aligned}$$

When ρ_v is a minimum relatively to S (*Fig.* 35.)

Then

$$\begin{aligned} \rho_v &= \frac{v}{4} \\ \overline{S} &= \pm\, \infty \\ \overline{Y} &= \mp\, \infty \\ \overline{i_Y} &= 180^\circ \end{aligned}$$

It will be seen, that, in the last case, the signs of Y have been reversed. The two branches of the curve cross each other and then approach the axis O Y and O Y_1 these being asymptotes to the curve.

The curve represented by equations (97) to (100) is the same as that of the contour assumed by a flexible sheet, or band, supported along two edges, and charged with a fluid, so that it takes the shape of a trough ; the length along the supported edges being taken as very great or the ends of the trough sealed in such a manner that the pressure of the fluid on the ends has no effect on the curve of the cross section.

Let Fig. 37 represent the curve of the section of such a trough, the sheet, or band, being supported at Y and Y_1 along its edges, and let Y Y_1 be the surface of the fluid.

Let the curve be referred to rectangular axes, O X and O Y call v the depth at the centre, and use letters and symbols for other factors as in the previous equations. Then, as the density of the fluid will not affect the form of the curve, assume unit density and unit width of the band, normal to the plane of the curve.

Then, the intensity of the normal pressure at any point in the curve $= x$

The intensity of the tension along the curve must be uniform, as, owing to its flexibility, it will take such a curved form as will make it uniform.

By (80), the tension T $= \rho_v v$ at the centre; and ρx at any other point, because v and x represent the pressure intensities at these points.

Therefore

$$T = \rho_v v = \rho x \quad \text{a constant} \qquad . \quad (117)$$

The horizontal component of the pressure on each half of the curve is equal to the pressure which would be exerted by the fluid acting on an immersed plane extending vertically from the surface to the depth v and is represented by the triangle adjoining Fig. 37.

Therefore the horizontal pressure on the half curve, between v and Y (calling that pressure H) is $H = \frac{v^2}{2}$.

In the same way, $\frac{x^2}{2}$ is the horizontal pressure on a portion of the curve between x and Y.

From this, the horizontal pressure between v and x $= \frac{v^2 - x^2}{2}$ and this is balanced by the difference between the tension T at v and the horizontal component of the tension at x which $= T \cdot \cos i$.

Substituting from (117), we have

$$\frac{v^2 - x^2}{2} = T - T \cdot \cos i = \rho_v v\,(1 - \cos i)$$

whence

$$\cos i = 1 - \frac{v^2 - x^2}{2\,\rho_v v} = 1 - \frac{v}{2\,\rho_v}\left(1 - \frac{x^2}{v^2}\right) \quad (118)$$

Now $\cos i_Y = 1 - \dfrac{v}{2\,\rho_v}$ when $x = 0$ and it will be seen that these equations are the same as (89) and (90).

As
$$\frac{v^2 - x^2}{2\,\rho_v\,v} = 1 - \cos i$$

$$x = \{v^2 - 2\,\rho_v\,v\,(1 - \cos i)\}^{\frac{1}{2}} \quad \{\text{the same as (92)}\}.$$

From this it follows that these curves are exactly the same as those of the bent spring, the line of the surface of the fluid corresponding to the line of application of the force.

A consideration of the problem of the flexible band enables us to readily obtain equations to other elements of the curves.

For example, the weight of the fluid over the half curve, from v to Y or, as we are considering a unit density, etc., the area enclosed between O Y and the curve, calling that area V is

$$V = T \cdot \sin i_Y = \rho_v\,v \cdot \sin i_Y$$

or

$$V = \rho_v\,v\,(1 - \cos^2 i_Y) \quad . \qquad (119)$$

or, by substitution

$$V = \rho_v\,v \left\{1 - \left(1 - \frac{v}{2\,\rho_v}\right)\right\}^{\frac{1}{2}} = \left(\rho_v\,v^3 - \frac{v^4}{4}\right)^{\frac{1}{2}} \quad (120)$$

When the ends of the curve become parallel, or the tangents become vertical

$$\rho_v = \frac{v}{2} \text{ and (120) becomes } V = \frac{v^2}{2} \qquad (121)$$

and, in this case, the vertical pressure, or area, between v and x becomes

$$\tfrac{1}{2}\,(v^4 - x^4)^{\frac{1}{2}} \qquad (122)$$

The distance Z (Fig. 37), from Y to the centre of gravity of the area between O Y and the curve, can be found thus :—The forces acting on the curve from v to Y are :—The tensions T and T (acting tangentially at v and Y) and the total normal pressure on the curve, which pressure we divide into its horizontal component H and its vertical component V Under these forces, the curve is in equilibrium. Taking moments round Y the moment of T at Y is 0 and equating the others to zero, we have

$$T v - V Z - \frac{2 H v}{3} = 0$$

therefore

$$V Z = T v - \tfrac{2}{3} H v \quad \text{and} \quad Z = \frac{T v - \tfrac{2}{3} H v}{V}$$

then, substituting values of T, H and V

$$Z = \frac{\rho_v v - \tfrac{1}{3} v^3}{(\rho_v v^3 - \tfrac{1}{4} v^4)^{\frac{1}{2}}} = \frac{6 \rho_v v - 2 v}{(36 \rho_v v - 9 v^2)^{\frac{1}{2}}} \qquad (123)$$

When the end tangents are vertical, substituting the value of $\rho_v = \frac{v}{2}$ (123) becomes

$$Z = \frac{v}{3} \quad . \quad . \quad . \quad (124)$$

In Fig. 37, the line of the resultant N of the normal pressure between v and Y must pass through the intersection of the two tangents T and T N is the resultant of H and V and therefore

$$N = (H^2 + V^2)^{\frac{1}{2}} = \left\{\frac{v^4}{4} + \left(\rho_v v^3 - \frac{v^4}{4}\right)\right\}^{\frac{1}{2}}$$

$$= (\rho_v v^3)^{\frac{1}{2}} \quad . \quad . \quad (125)$$

Also, as N is the resultant of T_Y and T_v

$$N = 2\left(T \cdot \sin \frac{i_Y}{2}\right) = \rho_v \, v \times 2 \sin \frac{i_Y}{2}$$

$$= \rho_v \, v \, (2 - 2 \cos i_Y)^{\frac{1}{2}} = \rho_v \, v \left\{2 - 2\left(1 - \frac{v}{2\,\rho_v}\right)\right\}^{\frac{1}{2}}$$

$$= \rho_v \, v \left(\frac{v}{\rho_v}\right)^{\frac{1}{2}} = (\rho_v \, v^3)^{\frac{1}{2}} \quad \text{(as before)}.$$

When the end tangents become vertical, $i_Y = \frac{\pi}{2}$

and

$$N \text{ becomes } = \left(\frac{v^4}{4} + \frac{v^4}{4}\right)^{\frac{1}{2}} = \frac{v^2}{\sqrt{2}}$$

or

$$N = T \times 2 \sin \frac{\pi}{4} = \frac{v^2}{2} \times 2 \times \frac{1}{\sqrt{2}} = \frac{v^2}{\sqrt{2}}$$

Instead of considering the curve as a section of a sheet, containing a fluid, it may be considered as representing the axis of an arch, cord or rib, in equilibrium under a distributed pressure over either its concave or convex side, and everywhere normal to the curve and of an intensity varying directly as its depth below the line $Y\,Y_1$ Under these circumstances, the ends of the curve may become crossed, as in Fig. 35, and the equations will still apply.

If the radius of curvature at v $(= \rho_v)$ is diminished until it equals $\frac{v}{4}$ equation (100), for the length of the half curve becomes

$$S = \frac{v}{2} \times F_1 \{\sin^{-1} 1\} = \infty \quad . \quad (126)$$

because

$$\sin^{-1} 1 = \frac{\pi}{2} \text{ and } F_1\left(\frac{\pi}{2}\right) = \infty \text{ by (73)}$$

Or, equation (95) becomes

$$S = \frac{v}{2}\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1 - \sin^2\phi)^{\frac{1}{2}}} = \frac{v}{2}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\cos\phi}$$

$$= \frac{v}{2} \cdot \log\tan\left(\frac{\pi}{4} + \frac{\frac{\pi}{2}}{2}\right)$$

$$= \frac{v}{2} \cdot \log\tan\frac{\pi}{2} = \infty \text{ as before.}$$

If ρ_v is diminished below $\frac{v}{4}$ the curve takes the form indicated in Fig. 36. It consists of an endless series of loops, and is then the ordinary Hydrostatic Arch.

CHAPTER VII.

THE DESIGN OF SECTIONS OF STRUTS.

If a strut or column *a b* is under the conditions shown in Fig. 38 (*a*), so that at *b* the longitudinal axis must remain in a fixed direction, but the end *a* is free to move and is unrestrained in any way, and if a load P is applied at *a* great enough to bend the strut, it will take a curved form such as *a b* The length S of the curve will be in accordance with equation (99) or (100).

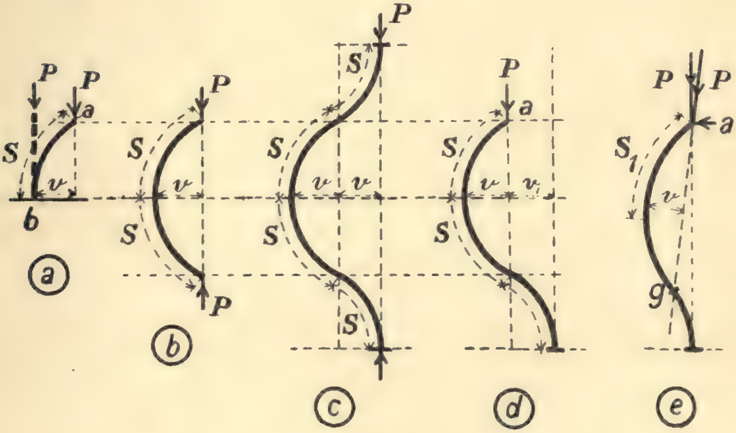


Fig. 38.

If, as in Fig. 38 (*b*), the axis of the strut is not fixed in direction, but is free to move at both ends, and a pair of equal and opposite forces P P are applied at its ends, it will become curved as indicated. The axis at the centre of the strut will remain parallel to the line P P and each half of the curve will therefore be similar to the curve of Fig. 38 (*a*). The equation for the value of S will then give the half length of the strut.

If, now, a strut, as in Fig. 38 (*c*), has the direction of its longitudinal axis fixed at both ends, and two forces PP applied, owing to symmetry of parts, all the portions marked S will be alike, and each will be similar to the curve of Fig. 38 (*a*). The equation for the value of S will then give the quarter length of the strut.

If the conditions were as shown in Fig. 38 (*d*), the strut being fixed at one end and the other end guided into a position where $v = v_1$ as indicated, the length L would equal 3 S Such conditions are never met with in practice, but in some instances a strut may be under conditions approaching those shown in Fig. 38 (*e*), with one end fixed and the other end guided so that it always lies on the line, produced, of the axis at the fixed end, and with the axis at the free end unrestrained from angular movement. Compared with Fig. 38 (*d*), the end *a* has moved over to the right through the distance v and the length S_1 from *a* to the point where the axis is parallel to *a g* is greater than S , and L will be somewhat less than $3\,S_1$.

It is impossible for a strut to be placed under more unfavourable conditions of support than as in Fig. 38 (*a*) unless P does not act along the axis, and the conditions cannot be more favourable than as in Fig. 38 (*c*).

Instead of continually referring to struts as having fixed ends, or free ends, etc., it is very convenient to give the different conditions of support Type Numbers, and taking the worst condition, Fig. 38 (*a*), as the basis and calling it Type No. 1, we have the following relations, the total length of the strut being, in all cases = L

For intermediate conditions the Type Numbers will be fractional.

These Type Numbers then give, for each particular case, the value of the constant in the equation for the length of the strut, whence we have

$$L = (\text{Type Number}) \times S \quad . \quad . \quad . \quad (127)$$

Condition of Support	Type	Length
Axis fixed in direction at one end, and unrestrained at other end, Fig. 38, (*a*)	No. 1	S
Axis free at both ends, but kept in line of action of P, Fig. 38, (*b*)	No. 2	2 S
Axis fixed at one end, other end free, but guided to position as in Fig. 38, (*d*)	No. 3	3 S
Axis fixed in direction at both ends, Fig. 38, (*c*) .	No. 4	4 S

The exact formula for the length and strength of a strut, under ideal conditions of loading, is given by equation (99). To make use of this equation it is possible to directly assign values to all of the varying factors except v that is, to the maximum deflection of the axis of the strut from the line of action of P It is therefore necessary to first find the maximum allowable value of v

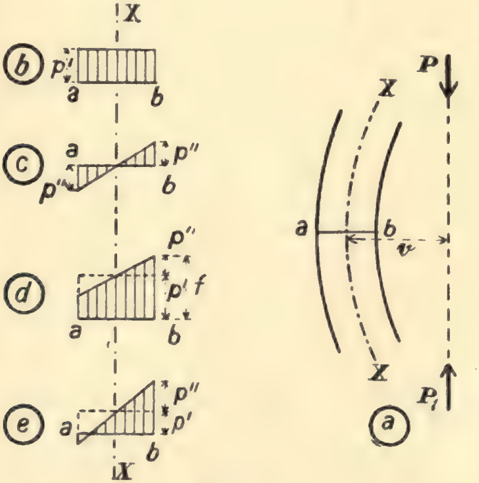


FIG. 39.

In Fig. 39, let (*a*) represent the portion of a strut where the deflection is greatest, and let P P_1 be the line of action of P On the section *a b* the stresses will consist of :—

(1) A direct stress, as in (*b*), equal to P distributed uniformly over $a\,b$ Calling the intensity of the stress p' and the sectional area A we have

$$p' = \frac{P}{A} \quad . \quad . \quad (128)$$

(2) Stresses, as in (*c*) produced by the bending moment $P\,v$ compressive on the side of the section nearest to P and tensile on the side furthest from P The maximum intensity p'' of these stresses, using the same notation as before, is

$$p'' = \frac{P\,v\,\delta}{I_0} \quad . \quad . \quad (129)$$

The combination of these two sets of stresses results in a distribution as shown in the diagrams (*d*) and (*e*). If p'' is less than p' the distribution will be as (*d*) the whole of the stresses being compressive, the maximum being $p' + p''$ and the minimum being $p' - p''$ If p'' is greater than p' the distribution will be as (*e*). On the side furthest from P the resulting stress will be tensile, and its numerical intensity $p'' - p'$ On the side nearest to P the stress will be compressive and its intensity $p' + p''$

If f is the maximum permissible compressive stress intensity, for which the calculations are to be made, we have

$$f = p' + p'' \quad . \quad . \quad (130)$$

and substituting from (129)

$$f = p' + p'' = p' + \frac{P\,v\,\delta}{I_0} = p' + \frac{P}{A} \cdot \frac{A}{I_0} \cdot v\,\delta$$

$$= p' + p' \cdot \frac{v\,\delta}{r^2} = p'\left(1 + \frac{v\,\delta}{r^2}\right) . \quad (131)$$

from which we have

$$p' = \frac{f}{1 + \frac{v\,\delta}{r^2}} \quad . \quad . \quad (132)$$

Transposing this equation, we get

$$\frac{v\,\delta}{r^2} = \frac{f}{p'} - 1$$

and therefore

$$v = \frac{r^2}{\delta}\left(\frac{f}{p'} - 1\right) \quad . \quad . \quad (133)$$

which can also be written

$$v = \frac{r^2\,\mathrm{A}}{\delta}\left(\frac{f}{p'\,\mathrm{A}} - \frac{1}{\mathrm{A}}\right) = \frac{\mathrm{I}_0}{\delta}\left(\frac{f}{\mathrm{P}} - \frac{1}{\mathrm{A}}\right) \quad (134)$$

This expression gives the *maximum allowable deflection* to keep f within its assigned limit, and in using the various equations it must be seen that the correct value of v is always inserted. It will be noticed that these equations relating to v, in a strut, differ considerably from those relating to beams or cantilevers, because in the case of struts the problem is not merely to find the value of v produced by any load, but to find the proportions of the strut which will prevent v increasing beyond the amount which will, in turn, produce the maximum assigned value of f.

There is no direct method of designing the section for a strut of a given length and to carry a given load owing to the various factors in the equations being so much dependent upon each other. The maximum allowable stress intensity f is the sum of the stress intensity p' (due to P) uniformly distributed over the section, and of the maximum stress intensity p'' caused by the bending moment due to the load not acting directly along the longitudinal axis. Now, p' cannot be determined until the area of the cross section is known, and p'' cannot be determined until the form and dimensions are determined. These are, however, the very things which have to be designed, and it is evident that the method of procedure must be a tentative one.

It is necessary to make trial calculations, based on some assumptions, and if the results are not satisfactory, to modify the assumptions until the desired results are obtained.

In using formulæ which do not take into account the form of the section, if the struts are small and formed of single rolled bars, it is possible to obtain the values indicated by the formula for A and r on assuming or settling upon certain relations between p' and p'' etc., and then to select, from a standard list of such sections, one having an area and radius of gyration as nearly as possible to those indicated.

If the section is to be built up the matter is more difficult, and it is then usual to first design a provisional section and insert the corresponding values of A and r etc., in the equations, when, if the derived value of P is too great or too small, the section is modified, further calculations made, and so on until it appears satisfactory.

If the form of the section is to be taken into account, still more care is required, but when dealing with rolled sections it is a comparatively easy matter to tabulate results for those which are used most frequently. For built up sections, of moderate dimensions, it is always advisable to consider the form of the section, and in the case of what may be termed large struts, and those in important structures, it is absolutely necessary to do so.

In practice, it is impossible to construct a strut with its axis perfectly straight, and to ensure that all other conditions are ideal. It is necessary, however, to first consider the effects of ideal conditions, and afterwards the effects of departures therefrom, and in doing so we will take, first of all, Isotropic sections, so that the Moment of Inertia I_0 is constant about all neutral axes.

As was shown in (111), Euler's formula does not represent the relative strength or load carrying capacity of any given strut, as its length increases, but merely the loads under which deflection would commence, at various lengths, under ideal conditions. In Fig. 40 *a b* represents the curve in accordance with this formula, for struts with ends free, Type 2.

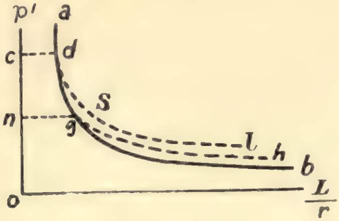


Fig. 40.

The equation

$$L = 2S = \sqrt{\frac{E\,I_0}{P}} \cdot \pi$$

applies to Type 2, and may be written

$$L = \sqrt{\frac{E\,I_0}{p'\,A}}\;\pi = \sqrt{\frac{E\,r^2}{p'}} \cdot \pi$$

wherefore

$$\frac{L}{r} = \sqrt{\frac{E}{p'}} \cdot \pi \quad \text{or} \quad p' = \frac{E\,\pi^2}{\left(\frac{L}{r}\right)^2} \qquad (135)$$

In the Fig. horizontal distances from O represent ratios of $\frac{L}{r}$ and vertical distances to the curve represent values of p' It will be seen that this curve depends upon no factors connected with the section of the strut, except the radius of gyration, and will therefore give the same results for struts of any magnitudes, so long as the ratios $\frac{L}{r}$ are similar.

If we wish to draw a curve of results given by the exact equations, and for varying ratios of $\frac{L}{r}$ it will be found practically impossible to calculate general values of p' corresponding to given values of $\frac{L}{r}$ but, if we assign values to f

and p' it is comparatively easy, for any given section, to find the corresponding value of $\frac{L}{r}$ and curves can be readily drawn.

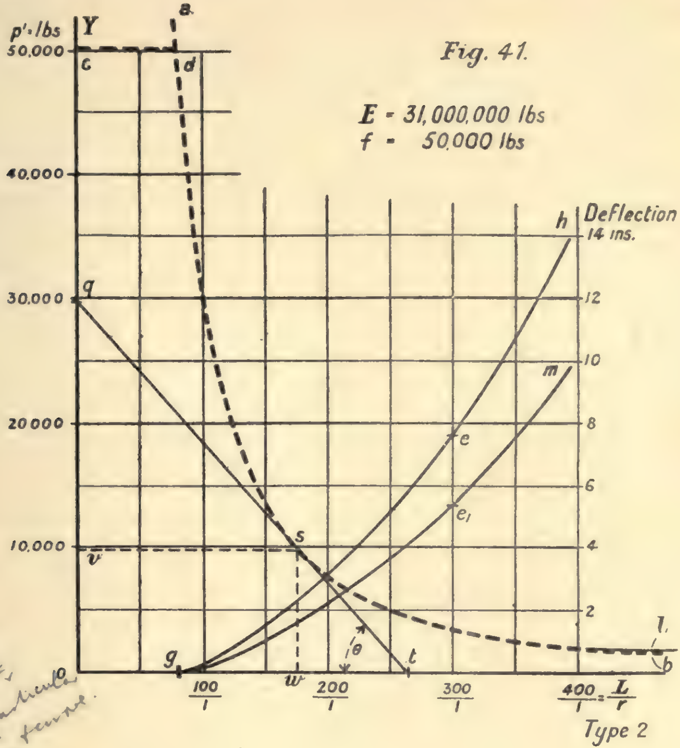
If, for example, we give f the value represented by the height oc (Fig. 40) it will be found that the curve will take the form cdl being horizontal from c to d and then curving from d to l and beyond. It does not extend above cd and the equations will not give any values for $\frac{L}{r}$ if p' is made greater than f and, of course, it cannot be greater. When $p' = f$; $\frac{L}{r}$ may be equal to, or anything less than represented by $\overline{cd}$, and the point d will lie on Euler's curve.

If we now take any other value for f say on we obtain a curve ngh being horizontal from n to g and then curving away from g As before, the point g will lie on Euler's curve. The portions of the curves dl and gh, etc., all lie above Euler's curve, and the greater the value of $\frac{L}{r}$ the greater the difference becomes.

It would be found that if cdl was the curve for any given section, it would not apply to another section having the same area and moment of inertia, if the form of the section differed. The line cd would remain the same, but the curved portion would vary and lie above or below dl according as the value of δ was less or greater. Even for an isotropic section, if not of circular form, the curve would be higher or lower at the end l if the direction of bending varied. In all cases the point d would lie on Euler's curve. The same remarks apply to any other curve such as ngh and so on.

It is therefore clear that, as the sections and permissible stresses vary, there is an infinite number of possible curves, and no single curve can represent exact values except for its own particular conditions. Presuming, however, that E always remains constant, no matter what the form of the section may be or what value may be assigned to f the points where these curves become horizontal, such as d and g etc., will all lie on Euler's curve. The curve given by Euler's formula is, p' = direct stress = P/A.

Fig. 41.



Type 2

ding
= 50000
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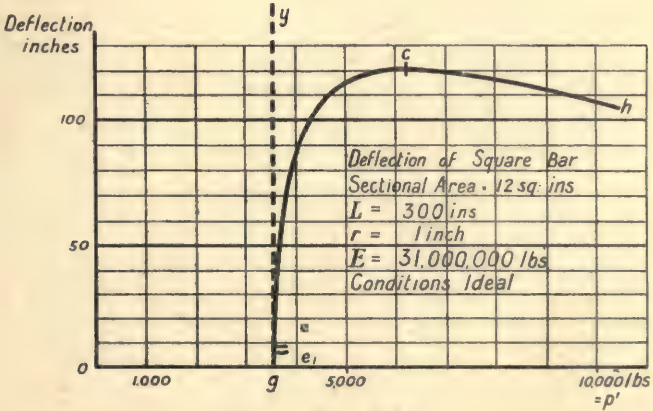


Deflection of Square Bar
Sectional Area = 12 sq. ins
L = 300 ins
r = 1 inch
E = 31,000,000 lbs
Conditions Ideal

Fig. 42.

in fact, nothing else than the equation to the locus of these points, as explained in (111).

Fig. 40 is not drawn to scale, the curves being separated for clearness. In Fig. 41, all are to scale. The dotted line *a b* is Euler's curve, for E = 31,000,000 lbs. The curve *c d l* is correctly drawn for the same value of E and for f = 50,000 lbs. per square inch, which is approximately the ultimate compressive strength of ordinary mild steel. It is also drawn for a section in which the ratio $\frac{\delta}{r} = \frac{1}{1}$ that is, the radius of gyration is equal to the distance from the axis to the outer fibre of the section. The ratio $\frac{1}{1}$ can never be actually reached, but it is the limit of the ratio and makes the curve *d l* lie at the maximum possible height above *a b* and all other possible curves must lie between them, for the same values of E and f

It will be seen that although *d l* lies above *a b* the difference, although the greatest possible, is very slight, so slight, in fact, that it is impossible to show any difference except near the extreme right. The portion *d b* of Euler's curve may be substituted for any of the true curves without sensible error, so far as matters relating to stresses on the sections are concerned.

In connection with matters of curvature and deflection, the same substitution is not permissible. As mentioned in Chapter II, attempts to determine deflections, by means of Euler's formula, lead to the conclusion that if p' is given any value above the curve *a b* by an amount however small, the strut will double up and collapse. Now, the values of p' represented by the true curve *d l* are above *a b* and yet the true analysis shows that, at these values, the forces and stresses are in equilibrium, although some bending and deflection have taken place. The curve *g h* (Fig. 41) shows the amount of deflection which would occur at the centre of a bar of steel, of the quality represented, and of a square section, if deflection took place parallel to the sides, and f not exceeding the assigned limit. The curve $g\,m$ shows the corresponding amounts of deflection if bending took place diagonally. For purposes of ease in calculation, r has been taken as 1 inch, and therefore the figures showing the ratios $\frac{L}{r}$ also represent, for that bar, the actual length in inches. When $r = 1$ the side of the square is $\sqrt{12}$ inches $= 3{\cdot}464$ inches.

Corresponding to each of these deflection curves, there should be a separate curve of values of p' both of them lying between $a\,b$ and $d\,l$ the curve corresponding to $g\,h$ lying higher than that corresponding to $g\,m$ because, as shown in (58), the square strut carries a greater load if it deflects parallel to the sides than if diagonally. The space between $a\,b$ and $d\,l$ is so small that the difference between these curves would be indistinguishable, and this illustrates the smallness of any errors due to the substitution of Euler's curve for the exact curves, but only when the conditions are ideal, and only in connection with the stresses on the section.

Another wrong conclusion regarding deflection, drawn from the substitution, is that after a straight strut begins to bend, it offers no increasing resistance to the force causing the bending, however much the bending may increase. This statement is quite incorrect, and from the instant that bending commences, the resistance commences to increase, and the rate of increase also increases, although it is true that at the point of commencement the rate of increase is nil.

Fig. 42 shows the deflection curve calculated for a bar of the same section as before, for a constant length of 300 inches, and for a gradually increasing load, P. Up to a load of 3400 lbs. there is no deflection. Then, according to the false deductions, the slightest increase of load should produce indefinite deflection, and the curve should rise vertically along the line $g\,y$ to any height. The true curve is correctly represented by $g\,c\,h$ It commences to rise vertically at g and at that point it is tangential to $g\,y$ It immediately, although at first slowly, begins to curve away to the right in the Fig. It reaches a maximum at c and then commences to fall. The curve is at first concave towards O X Afterwards the curvature reverses, and the curve continually approaches O X which is an asymptote to it.

For any specific amount of deflection, except the maximum, there are two corresponding loads or values of p' For example, if v equals 108 inches, p' may be 4400 lbs., or 9750 lbs., these being, first, the load which will bend the bar to the form of a simple bow as in Fig. 31, and, second, the load required to bring the bar to the form of Fig. 35, with the ends crossed, v being the same in both cases. Of course, in ordinary struts the range of practical deflection is small, and the greater portions of these curves represent deflections for values of f much above the elastic limits of materials, and are, for that reason, and in that connection, of no value. At the same time, with high ratios $\frac{L}{r}$ and long lengths, struts will bend to an extent much greater than generally supposed, before the ultimate stress intensities are reached, and these curves are necessary to illustrate the correct principles of design. For flexible springs which can be bent to a great extent, without being strained over the elastic limit, they truly represent the facts, if the springs are arranged so that their own weight does not affect the curvature.

In Fig. 42, the curve $g\,c\,h$ is the deflection curve for the bar, in whatever direction it may bend, whether parallel to the sides, or diagonally, or in any other direction. As the section is isotropic, deflection always commences under the same load at g whatever the direction of bending may be. If the bar deflects parallel to the sides, the ultimate intensity of $f = 50{,}000$ lbs. is reached at the point e on the deflection curve, while if it deflects diagonally, f is reached at the point e_1 The amount by which the point e lies to the right of e_1 (Fig. 42) is the same as the difference which there would be in the height of the two curves of p' corresponding to $g\,h$ and $g\,m$ in Fig. 41, at $\frac{L}{r} = \frac{300}{1}$. Also the perpendicular distance from gy to e or e_1 is the amount by which the corresponding curve would lie above Euler's curve in Fig. 41 $\left(\text{at ratio } \frac{L}{r} = \frac{300}{1}\right)$, being, of course, plotted to the corresponding scale. Fig. **42** is to a larger scale than Fig. 41, so far as values of p' are concerned. This further illustrates the smallness of the differences between Euler's curve and the true curve. So far as the linear amount of deflection is concerned, the two points e and e_1 in Fig. **42** are the same as those bearing the same letters in Fig. 41.

An exact idea of the smallness of the errors resulting from the use of Euler's curve for the true curve is given by the following table, which, for various values of p' gives the corresponding values of $\frac{L}{r}$ obtained by the application of Euler's formula and the exact equations respectively. The comparison is made for the same values of E and f as those

STRUTS OF TYPE 2, UNDER IDEAL CONDITIONS.

$E = 31{,}000{,}000$ lbs. $f = 50{,}000$ lbs. $\frac{\delta}{r} = \frac{1}{1}$.

—	Values of $\frac{L}{r}$ obtained by		Differences
Direct Stress Intensity, p^1	Euler's Formula	Exact Equations	Per Cent.
lbs. per sq. in.			
20,000	$\frac{123 \cdot 68}{1}$	$\frac{123 \cdot 70}{1}$	0·02 %
10,000	$\frac{174 \cdot 92}{1}$	$\frac{174 \cdot 99}{1}$	0·04 %
5,000	$\frac{247 \cdot 37}{1}$	$\frac{247 \cdot 58}{1}$	0·085 %
2,000	$\frac{391 \cdot 12}{1}$	$\frac{392 \cdot 04}{1}$	0·23 %

for which the curves are drawn. The value f is the ultimate compressive strength of ordinary mild steel. The values in the column headed "exact equations" are for $\frac{\delta}{r} = \frac{1}{1}$ and the corresponding differences are therefore the greatest possible for any form of section but always under ideal conditions of loading.

This shows that in a strut of Type 2, with a ratio of $\frac{L}{r}$ equal to about $\frac{250}{1}$, and which is a very extreme ratio, the difference between the results is only 0·08 per cent. in the length of the strut corresponding to any given section, while for equal lengths the derived values of p' differ by a less percentage. Even with an almost impossible ratio of $\frac{L}{r}$ equal to about $\frac{400}{1}$ the difference does not exceed 0·23 per cent. With lower values of f the differences are still less.

Euler's curve is the same as the curve obtained by taking a rectangular hyperbola, referred to its asymptotes as axes, and squaring all the horizontal ordinates.

Thus, calling vertical ordinates y and horizontal ordinates x the equation to the hyperbola is

$$y\,x = \text{(a constant)}; \quad \text{therefore} \quad y = \frac{c}{x}$$

For Euler's curve, we have by (135),

$$p' = \frac{E\,\pi^2}{\left(\frac{L}{r}\right)^2} \qquad \text{in which } E\,\pi^2 \text{ is a constant.}$$

Then p' corresponds to y and $\frac{L}{r}$ to x, and substituting, we get

$$y = \frac{\text{(a constant)}}{x^2} \qquad \text{or} \qquad y = \frac{c}{x^2}$$

In Euler's curve the tangent at any point is

$\tan\theta = -\dfrac{dy}{dx}$ the sign of dy being – because y decreases as x increases, then

$$-\frac{dy}{dx} = \frac{2c}{x^3} = 2 \cdot \frac{c}{x^2} \cdot \frac{1}{x} = \frac{2y}{x}$$

Now, referring to Fig. 41, at any point s in the curve, qt is the tangent, produced to cut O Y and O X ; $Ow = x$ and $ws = y$

Then

$$\overline{wt} = \frac{\overline{ws}}{\tan\theta} = \frac{y}{\left(\frac{-dy}{dx}\right)} = y \cdot \frac{x}{2y} = \frac{x}{2}$$

and

$$\overline{Ot} = \overline{Ow} + \overline{wt} = x + \frac{x}{2} = \frac{3x}{2} = x \times 1\tfrac{1}{2}$$

Again,

$$\overline{vq} = \overline{vs} \cdot \tan\theta = x \cdot \tan\theta = x \cdot \frac{2y}{x} = 2y$$

and

$$\overline{Oq} = \overline{Ov} + \overline{vq} = y + 2y = 3y$$

Thus we find that for any point on the curve the tangent produced cuts the axis of O Y at a distance from O equal to three times the vertical ordinate, and cuts the axis of O X at a distance from equal to one and a half times the horizontal ordinate. These ratios are useful in drawing curves, or checking them when drawn.

In the rectangular hyperbola the corresponding distances are $2y$ and $2x$ respectively.

There is no real difficulty in applying the exact equations to the solution of a strut problem, but that known as Euler's formula is simpler and does not necessitate the use of tables of Elliptic Functions, which are not commonly available.

From the above it is quite clear that, for ideal conditions of loading, equation (110) may be used to determine the relations between $\frac{L}{r}$ and p' and that the errors due to the substitution are quite inappreciable. . . . (136)

At the same time the limitations of the equation must not be lost sight of, and it should not be used for any purpose connected with curvature and deflection.

In the practical construction and loading of struts the conditions can never be ideal. There are many causes of departure from ideal conditions, and the most important of these are :—

1. Eccentric application of the load at the end of the strut.

2. Inaccuracies of workmanship, causing a want of alignment in the longitudinal axis.

3. Want of homogeneity in the material, causing a variation in the value of the modulus of elasticity.

No explanation is required of No. 1. It is practically impossible, with ordinary methods of applying a load to a column or strut, to ensure that the centre of pressure coincides with the axis.

With regard to No. 2, it is well known that there is difficulty in straightening rolled bars. If a strut is built up of separate bars riveted together, even if each of the separate bars be carefully straightened, the expansion of the rivets, when being closed, sets up internal stresses in the material, and therefore causes curvature. No matter how carefully and systematically the riveting is done, there are almost certain to be some slight irregularities in the strut when completed. The effect of this is to cause a deviation of the axis from the straight line, either locally or throughout.

The effect of No. 3 is to cause irregularities in the neutral axis, not in a geometric sense, but to produce a want of coincidence between the geometric axis and the neutral axis, considered in a physical sense, as explained in (47), and which is the real neutral axis. Here, again, the effect is to cause a deviation of the axis from the line of pressure.

There are other minor causes of irregularity; for example, there may be initial internal stresses in the material. As a general rule these will not be sufficient to cause any irregular deflection under load, and unless they were so great that the additional stress, due to the bending, produced an intensity at any point greater than the elastic limit, there could be no irregularity; but if the elastic limit was exceeded, a permanent set would result. Even a change of temperature is sufficient to cause curvature, although probably always of a temporary character.

Other disturbing causes are, deflection due to the weight of the strut itself, which, unless the strut is vertical, will always exist and should be considered; also temporary causes, such as wind pressure, etc., but such causes are generally investigated separately.

It will be seen that the effect of any kind of disturbance is to cause a departure of the centre line of pressure or application of the external force P from the axis of the section, or vice versa, and the effect on the resulting stresses is similar in all cases.

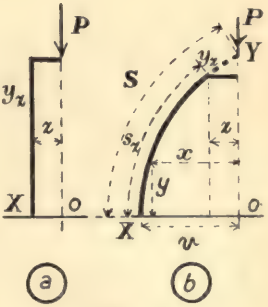


FIG. 43.

Consider, now, a straight strut X y_z of Type 1, as Fig. 43, with the load P applied along the line P O at a distance z from y_z The strut will bend and assume a form as in (*b*), and let it be assumed that the distance z (from the line of P to y_z) is the same in both (*a*) and (*b*). Let the curve be referred to the rectangular axes O X and O Y At any point in the curve there is a bending moment P x and the curve is therefore the same as a portion of that shown in Fig. 31. If produced, as shown in (*b*), to Y it is exactly the same as X Y in Fig. 31.

The length of the completed curve S is given by equation (99). The length of the curve s_z is that of the portion of the complete curve S between the limits of $x = z$ and $x = v$ and will therefore be obtained by substituting these values for s and x and v in equation (97), which then becomes

$$s_z = \mathrm{K} \cdot \int_0^{\phi_z} \frac{d\,\phi}{(1 - \sin^2\theta \cdot \sin^2\phi)^{\frac{1}{2}}}$$

$$= \mathrm{K} \times \mathrm{F}\,(\theta, \phi) = \mathrm{K} \times \mathrm{F}\left\{\left(\sin^{-1}\frac{v}{2\,\mathrm{K}}\right),\ \left(\cos^{-1}\frac{z}{v}\right)\right\} \quad (137)$$

where

$$\mathrm{K} = \sqrt{\frac{\mathrm{E\,I_0}}{\mathrm{P}}}$$

$$\theta = \text{the angle whose sine is } \frac{v}{2\,\mathrm{K}}$$

and

$$\phi = \text{the angle whose cosine is } \frac{z}{v}$$

The value of s_z can be obtained from this equation, for any given section, by inserting the corresponding values of E, $\mathrm{I_0}$, P and z and also the value of v obtained by equation (133). As in the case of the exact equations for the column under axial loading, the solution of equation (137) involves the use of Elliptic Functions. In this case they are not the Complete Functions, but those of ϕ_z which is not $\frac{\pi}{2}$ but something less. As explained in Chapter V., it is practically impossible to tabulate values of F for all values of ϕ and, for this reason, equation (137) is too troublesome for ordinary use.

A simple and close approximation to the value of s_z can be made. An inspection of Fig. 44 will show that the length

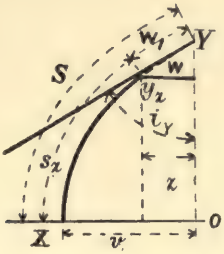


FIG. 44.

$\overline{X y_z}$ is equal to the difference between $\overline{X\,Y}$ and $\overline{y_z\,Y}$ or (calling $\overline{y_z\,Y} = W_1$)

$$s_z = S - W_1 \quad . \quad . \quad (138)$$

The curvature of X Y varies as x and therefore at the point Y the curvature is nil, and the curve merges into a straight line. Now the tangent Y T coincides in direction with the curve at the point Y and therefore the two not only come into contact at that point, but absolutely coincide with each other. For some distance from Y (towards X) the curve and the tangent differ very little from each other, and the length of the tangent subtending z (which call W) may be substituted for the length of the portion W_1 of the curve. The angle T Y O between the tangent and O Y is i_Y of the equations, and, by (101), $i_Y = 2\,\theta$ therefore, approximately,

$$W = \frac{z}{\sin i_Y} = \frac{z}{\sin (2\,\theta)}$$

or, as θ and i_Y are, in practical struts, always exceedingly small, we may substitute $2 \cdot \sin \theta$ for $\sin (2\,\theta)$. For ordinary amounts of deflection $\frac{1}{2}°$ for θ and $1°$ for i_Y are very large values, and $\sin \frac{1}{2}° = \cdot 008\ 726\ 5$, while

$$
\begin{array}{lrcl}
 & \text{Sin } 1° \text{ is} & & 0{\cdot}017\ 452\ 5 \\
\text{now,} & 2 \times \sin \frac{1}{2}° & = & 0{\cdot}017\ 453\ 0 \\
\hline
 & \text{the difference being} & & 0{\cdot}000\ 000\ 5
\end{array}
$$

involving only a fractional error of about $\frac{1}{35{,}000}$ part in one factor of the equation. We then have, very nearly,

$$W_1 = W = \frac{z}{\sin i_Y} = \frac{z}{2 \cdot \sin \theta} \quad . \quad (139)$$

Substituting this in equation (138), we get

$$s_z = S - \frac{z}{2 \cdot \sin \theta} \quad . \quad (140)$$

where, by (96), $\sin \theta = \dfrac{v}{2K}$

therefore

$$2 \cdot \sin \theta = \frac{v}{K}$$

and

$$s_z = S - \frac{z}{v} K = K\left(\frac{\pi}{2} - \frac{z}{v}\right) \quad . \quad (141)$$

and substituting value of K, from (83)

$$s_z = \sqrt{\frac{E\ I_0}{P}}\left(\frac{\pi}{2} - \frac{z}{v}\right) \quad . \quad (142)$$

A deviation z due to the first cause referred to, namely, eccentric application of the load will be relatively more serious as r decreases, but an increase in L will not increase the difficulty of making it concentric. Therefore we should expect the effect of such deviation to have a minimum value for $\frac{L}{r} = 0$ and to increase with the value of $\frac{L}{r}$

Any deviation z due to the second cause (inaccuracy of workmanship in aligning the strut) will probably be relatively greater as the length of the column increases, and its effect will also be greater as r decreases, and therefore will relatively increase as $\frac{L}{r}$ increases, either by increase of L or decrease of r being zero when L $=$ 0

With regard to variation in E we should expect to find that as the sectional area increased, the probability of the average value of E being better maintained relatively to any neutral axis would also increase, owing to the surface of the section being greater, and it would therefore generally increase with r As the length increases, the number of cross sections, or layers into which the column could be cut by cross sections increases, and the probability decreases that any variations in E will take place in the same direction in them all, and if these directions vary, the resulting effect on the column as a whole will decrease. Therefore the amount of departure of P from the axis, due to variation in E, will generally decrease with increase of r as well as with L . As in the ratio $\frac{L}{r}$ one of these values is the numerator and the other the denominator, it may be taken that the proportionate net result is not much affected by the value of $\frac{L}{r}$

As a combination of effects due to these three causes, we have the probabilities that as $\frac{L}{r}$ increases from zero to a maximum, the effects of

No. 1, increase from a minimum to a maximum,
No. 2, increase from zero to a maximum,
No. 3, remain approximately constant.

The net result being an increase from a minimum value to a maximum, and the rate of increase being greater when the value of $\frac{L}{r}$ is small than when larger.

Now, equation (137) gives results for the condition that the amount of the initial deviation of P remains constant throughout the whole length of the strut, and is unaffected by an increase in the length. The partially empirical equation (142), however, introduces a variation exactly in the direction required to meet the above conclusions regarding the relative variations. Compared with equation (137), it makes no appreciable difference in the results for the higher ratios of $\frac{L}{r}$ but for the lower values it decreases the difference between the resulting value of L for any given section and load, and that indicated by the equation for ideal conditions. The equation (142) therefore meets the requirements.

If we now take equation (142) and substitute values of v from (133), we get

$$s_z = \sqrt{\frac{E\,I_0}{P}}\left\{\frac{\pi}{2} - z\frac{\delta}{r^2}\frac{1}{\left(\frac{f}{p'} - 1\right)}\right\} \quad . \quad (143)$$

or

$$s_z = \sqrt{\frac{E\,I_0}{P}}\left\{\frac{\pi}{2} - \frac{z}{r}\cdot\frac{\delta}{r}\left(\frac{p'}{f - p'}\right)\right\}$$

therefore

$$s_z = \sqrt{\frac{E\,I_0}{P}}\left\{\frac{\pi}{2} - \frac{z}{r}\cdot\frac{\delta}{r}\cdot\frac{p'}{p''}\right\} \quad . \quad (144)$$

Before these equations can be used it is necessary to determine a value for z The chief cause for the existence of z is probably that already called No. 3 ; a variation in E From (46), it will be seen that the result is a movement of the neutral axis through a distance being a certain proportional part of r For the sake of simplicity we may also express the corresponding variations due to all causes as a fraction of r Then, calling that fraction c we have $z = c\,r$ In equation (144) the factor $\frac{z}{r}$ then becomes $\frac{c\,r}{r} = c$ and, substituting this

$$s_z = \sqrt{\frac{E\,I_0}{P}}\left\{\frac{\pi}{2} - c\cdot\frac{\delta}{r}\cdot\frac{p'}{p''}\right\} \quad . \quad (145)$$

In the earlier column formulæ there is always a factor, or its equivalent, which has to provide for all causes of departure from ideal conditions, and the value of which has to be determined empirically from the results of tests. As we have seen, it has only been possible to assign to that constant values which will bring the result low enough to agree with the lowest of the observed results. Here, we have separated the factors, and now have one $\frac{\delta}{r}$ which depends on the form of the section and allows for the relative efficiency or non-efficiency of the design, and another factor c as a constant in the equation, the value of which is to be obtained, partly from the results of tests of the material which show the probable variation of E and partly from observation of the degree of accuracy attained in construction, or from the specified limits of variation. The factor $\frac{\delta}{r}$ in combination, as it is, with $\frac{p'}{p''}$ is a natural one resulting from the design. The other factor c although it appears as a constant in the equation, is connected by multiplication with the factor $\frac{p'}{p''}$ and this serves to give the necessary variation to the resulting effects of initial deviations, as $\frac{L}{r}$ varies. This factor is partially empirical, but, as has been shown, any errors due to its use are exceedingly small and negligible.

Under practical conditions the ultimate strength of a strut therefore depends upon the ratio of $\frac{\delta}{r}$ and as a decrease in that ratio increases the strength, the more nearly r is equal to δ the greater is the strength.

As an increase in r increases the strength, the material should be distributed so as to attain this condition as far as practicable. Of all sections the circular form gives the greatest value of r for a given area, having regard to the fact that r has to be measured in any direction

In the same way a relative decrease in δ increases the strength, and again, the circular form is the most advantageous as it makes δ a minimum.

A hollow circular section gives a lower ratio of $\frac{\delta}{r}$ than a solid one. For a given area the hollow circular section gives better results as the diameter is increased, but then the thickness of metal decreases and questions of local flexure have to be considered.

A circular section is not often permissible, and with any other form the value δ is not the same in all directions. The ratio $\frac{\delta}{r}$ must then be taken in the direction in which deflection or an initial deviation of P from the axis has the greatest effect.

In designing sections, as already explained, projecting parts of comparative narrowness must be avoided as much as possible, so as to make δ as small as practicable in relation to r and to avoid extreme variations in δ

In Fig. 45, a number of curves of values of p' corresponding to values of $\frac{\delta}{r}$ are shown for the practical conditions, in relation to the curve for ideal conditions.

These curves are in accordance with equation (145), and the effect of various values of $\frac{\delta}{r}$ is shown. Separate curves are drawn for $\frac{\delta}{r} = \frac{1}{1}, \frac{1 \cdot 5}{1}, \frac{2}{1}, \frac{3}{1}$ and $\frac{4}{1}$ respectively. It is impossible to make the ratio less than $\frac{1}{1}$, and $\frac{4}{1}$ is a general extreme ratio for the less economical forms of section.

In drawing the curves the value of the constant c has been taken at $\frac{1}{10}$, representing an initial deviation of the centre of pressure from the physical axis of the column to the extent of about one tenth of the radius of gyration, but increasing with the length of the strut, as already explained. There is no direct method available of arriving at the most

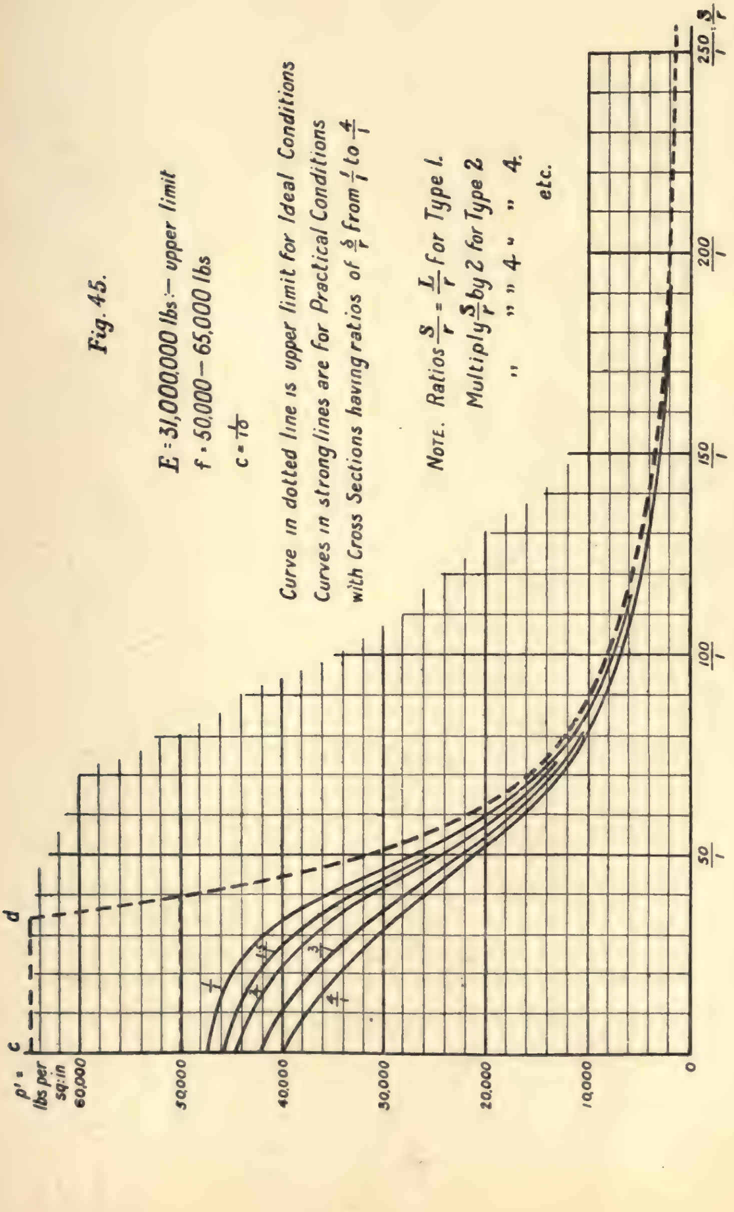


Fig. 45.
$E = 31{,}000{,}000$ lbs:— upper limit
$f = 50{,}000 - 65{,}000$ lbs
$c = \frac{1}{10}$
Curve in dotted line is upper limit for Ideal Conditions
Curves in strong lines are for Practical Conditions
with Cross Sections having ratios of $\frac{s}{r}$ from $\frac{1}{1}$ to $\frac{4}{1}$
Note. Ratios $\frac{S}{r} = \frac{L}{r}$ for Type 1.
Multiply $\frac{S}{r}$ by 2 for Type 2
" " " 4 " " 4.
etc.
p' = lbs per sq:in
60,000
50,000
40,000
30,000
20,000
10,000
0
c
d
$\frac{1}{1}$
$\frac{1\frac{1}{2}}{1}$
$\frac{2}{1}$
$\frac{3}{1}$
$\frac{4}{1}$
$\frac{50}{1}$
$\frac{100}{1}$
$\frac{150}{1}$
$\frac{200}{1}$
$\frac{250}{1} : \frac{S}{r}$

representative value of c but $\frac{1}{10}$ is the value which appears to be indicated by the majority of experiments for steel and iron. In the process of designing struts this value must be decided upon, and some regard must be had to the class of material and workmanship to be expected. If the conditions are not very good, the constant must be increased accordingly to, say, $\frac{1}{8}$ or more. There appears to be a need of further experiments to supply more exact data bearing on this point.

We saw, by (59), that for an isotropic section the direction of greatest effect of a deviation of P is in the direction of greatest δ

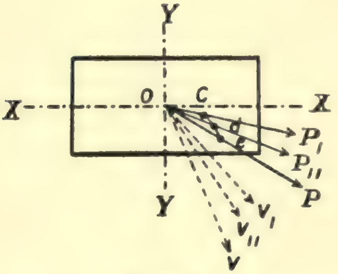


FIG. 46.

When the section is not isotropic, we saw by (62) that deflection does not take place in the plane of action of the external force. Consider such a section as in Fig. 46. If there is an initial deviation of P to the extent of O C in the direction of O P_1 then P_1 will represent a force tending to cause bending. This bending will not take place in the direction of O P_1 but in another direction such as O V_1 When this deflection commences the distance O C will increase, but C will not move along O P_1 but in a direction parallel to O V_1 and take up a new position d The deflecting force now acts in the direction O P_{11} and deflection will continue in another direction O V_{11} the point d changes to e and so on until the parts are in equilibrium. The loci of the points C and d and e etc., form a curve having a tangent at C parallel to O V_1 The equations to these curves of movement are too intricate for general use, and in practice we are concerned only with the final direction and amount of deflection when equilibrium occurs.

If e in Fig. 46 represents the final position of the action of P then O P is the direction of the deflecting force and O V the direction of the deflection. The bending moment in the direction O P is $P \times \overline{Oe}$ and the component in the direction of O V is $P \times \overline{Oe} \times \cos(e\,O\,V)$ but as the angle $e\,O\,V$ will not, under ordinary conditions, be large enough to cause a great difference between

$$P \times \overline{Oe} \quad \text{and} \quad P \times \overline{Oe} \times \cos(e\,O\,V)$$

such difference may be neglected. It will affect the final result to an inappreciable extent.

To find the required direction of O V for greatest effect of initial deviation of P (represented by the angle λ in Fig. 47), it is necessary to make use of equation (145). Equation (60) does not apply because, as just explained, the direction of action of the deflecting force does not remain constant. We require to find the angle λ which makes s_z a minimum, and must therefore express I_0 and δ in terms of λ By equation (43) we have

$$I_0 = I_{OX} \cdot \cos^2\lambda + I_{OY} \cdot \sin^2\lambda$$

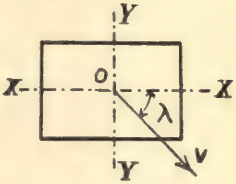


FIG. 47.

The forms of sections vary so much that no general expression can be given for δ in terms of λ and it must be found for each particular case.

Then, by equation (145), on eliminating the constant factors, s_z is a minimum when

$$\sqrt{I_0} - \left(\frac{2\,c\,p'\sqrt{A}}{\pi\,p''}\right) \cdot \delta = \text{a minimum},$$

and substituting the above,

$$(\mathrm{I}_{\mathrm{OX}} \cdot \cos^2 \lambda + \mathrm{I}_{\mathrm{OY}} \cdot \sin^2 \lambda)^{\frac{1}{2}} - \left(\frac{2\,c\,p' \sqrt{\mathrm{A}}}{\pi\, p''}\right) \cdot \delta$$
$$= \text{a minimum} \quad . \quad . \quad (146)$$

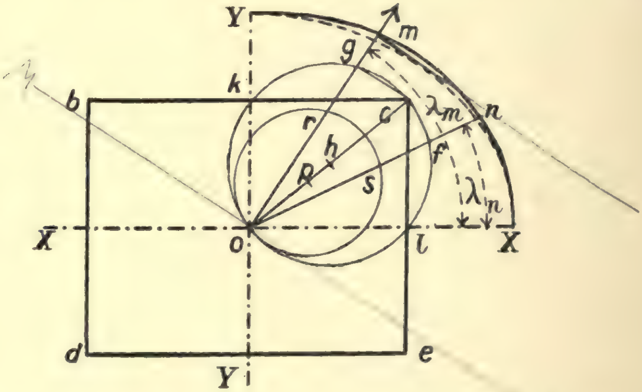
Differentiating these, we have, when s_z is a minimum,

$$\left.\begin{aligned} d\,(\mathrm{I}_{\mathrm{OX}} \cdot \cos^2\lambda + \mathrm{I}_{\mathrm{OY}} \cdot \sin^2\lambda)^{\frac{1}{2}} &= \left(\frac{2\,c\,p' \sqrt{\mathrm{A}}}{\pi\, p''}\right) \cdot d\,\delta \\ \therefore = \frac{(\mathrm{I}_{\mathrm{OY}} - \mathrm{I}_{\mathrm{OX}}) \sin\lambda \cdot \cos\lambda}{(\mathrm{I}_{\mathrm{OX}} \cdot \cos^2\lambda + \mathrm{I}_{\mathrm{OY}} \cdot \sin^2\lambda)^{\frac{1}{2}}} &= \left(\frac{2\,c\,p' \sqrt{\mathrm{A}}}{\pi\, p''}\right) \cdot d\,\delta \\ \text{or} \quad \frac{(\mathrm{I}_{\mathrm{OY}} - \mathrm{I}_{\mathrm{OX}}) \sin\lambda \cdot \cos\lambda}{\sqrt{\mathrm{I}_{\mathrm{O}}}} &= \left(\frac{2\,c\,p' \sqrt{\mathrm{A}}}{\pi\, p''}\right) \cdot d\,\delta \end{aligned}\right\} (147)$$

λ

$\frac{d\lambda}{d\delta}$

It is a comparatively easy matter to find approximately the angle λ by graphical methods. An example is given in Fig. 48. The contour of the section is a rectangle *b c d e* and



Om ≟ I mn

FIG. 48.

X X and Y Y are the principal axes. To any suitable scale mark off O X equal to $\sqrt{\mathrm{I}_{\mathrm{OY}}}$ and O Y equal to $\sqrt{\mathrm{I}_{\mathrm{OX}}}$ and draw the quadrant of an ellipse between these two points, as shown by the dotted line. Then, by equation (43), any line such as O n or O m drawn from O perpendicular to tangents to the ellipse, represent values of $\sqrt{I_0}$. Values of δ are represented by lines radiating from O, such as Of or O c or O g to the points where perpendiculars from c meet them. In this case, as any line such as O g with the corresponding perpendicular forms a right angled triangle with O c as the hypothenuse, the locus of all points such as g or f forms a circle with radius O h centre at h midway between O and c and passing through k, c, l and O. These values of δ multiplied by the value $\frac{2cp'\sqrt{A}}{\pi p''}$ from equation (146), and marked off from O in the same direction, give points forming another circle passing through O with centre p such that the radius $\overline{Op} = \overline{Oh}\,\frac{2cp'\sqrt{A}}{\pi p''}$ Now, along any vector such as O m the length $\overline{Om}$ represents the value of the portion of the left hand side of equation (146), before the $-$ sign, and $\overline{Or}$ represents the value of the portion after the $-$ sign, while $\overline{rm}$ is the value given by the equation. It is then an easy matter to find by trial the direction of O m which gives $\overline{rm}$ a minimum.

These equations are, of course, not absolute, but any errors due to the substitutions made are negligible. The graphic method is therefore also approximate, but it will be found that an error of a considerable amount in the value of λ will make little difference in the final result, as the value of rm alters slowly at first on each side of the position of mimimum, and an approximate determination of the angle λ is therefore sufficient.

For other forms of section if the contour is formed of straight lines the loci of points at the extremities of vectors will always lie on circles passing through O and the extreme points of the contour, as in Fig. 49,

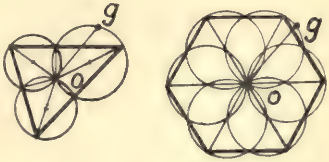


FIG. 49.

and these are easily drawn. When the section is irregular it may be necessary to draw all these circles, in order to find the minimum difference between the two values of equation (146).

The angle λ is affected by the values of p' and p'' and A. A change in any one of these will alter λ and as p'' becomes less towards the ends, or points of contrary flexure, in a strut, the direction of bending is not the same near these points as at the centre or point of maximum deflection. The deflection curve is not a plane curve, but a form of compound helix. It is, however, only necessary to consider the maximum values.

The above also shows that, if the section is not isotropic, when a load is applied, bending will not continue in the same direction as that in which it commences, but will alter as P increases, unless by chance the deflecting force acts in the plane of one of the principal axes.

It must be carefully noted that the above remarks as to the direction of maximum $\frac{\delta}{r}$ are on the assumption that the strut is under the same conditions or has the same Type Number in all directions, and not, as is often the case, say free ended or of Type 2 in one direction and approaching to Type 4 at right angles to it, as, for example, a strut with pinned ends. In such a strut the diameter and also the Moment of Inertia are generally greatest in the direction of the axis of the pin and then the direction of equivalent maximum $\frac{\delta}{r}$ will nearly always be practically normal to the pin axis. If the Moment of Inertia is least in the direction of the pin axis, the equivalent of maximum $\frac{\delta}{r}$ may lie in any direction and must be specially determined.

There is another matter which may be taken into account when considering a column having a large value of the ratio $\frac{L}{r}$ We saw, by (130), that with a certain degree of bending tensile stresses appear on the convex side of a bent strut. If E is the same for both tension and compression the equations and curves are not affected by the change in the nature of the stresses. If E is greater for tensile stresses the portion of the curve of p' (after tension appears) will be raised, and if E for tension is less the curve will be lowered. Steel is the only material from which struts of any considerable magnitude are, at present, made, and as for that material and for wrought iron E is generally slightly higher for tension, the right hand portions of the curves in the diagrams for practical conditions should be raised, but the difference is so slight for these materials that it is not necessary to take it into account. So far as the curve for ideal conditions is concerned, it is of no practical value, but is necessary for the analysis.

On the concave side of the bent strut the total stress intensity is, by (130), $p' + p''$ and on the convex side $p' - p''$ On the convex side p'' is tensile, and if it exceeds p' the net result is a tensile stress intensity of the numerical value $p'' - p'$ This tensile stress commences at the point where $p' = p''$ and therefore, where p' and p'' are each one half of f Thus, in Fig. 45, where the height from O X to the curve is half of its maximum height (which equals f) the resulting stress intensity at the extreme edge of the section on the convex side is nil, and for greater values of p' there will be tensile stresses on that side. This applies whether we take the curve for ideal loading, or for the practical conditions.

In Fig. 50, let $a\,b$ be the section of a strut under these conditions (when p'' exceeds p') and let it be symmetrical about the axis X X so that $o\,a = o\,b$ and let $a\,c = b\,d$ be the intensity p' of the direct compressive stress. If the modulus of elasticity E has the same value for tensile and compressive stresses, the value of p'' will be the same on both sides of the axis, and $\overline{d\,e} = \overline{c\,h}$ The portion $\overline{a\,h}$ of $\overline{c\,h}$

is $p'' - p'$ and represents the intensity of the tensile stress at the edge of the section.

If, now, E is not the same, but, as is usually the case, greater for tension than compression, a given strain (such as represented by $\overline{ah}$) will correspond to a greater stress intensity when in tension, and therefore the stresses represented by the area of the small triangle $a\,h\,l$ have greater values in their moments about the axis than those represented by a corresponding portion of $g\,e\,d$ which represents compressive stresses. As the sum of the bending stresses on one side of the physical axis must balance those on the other, it follows that the triangle $g\,e\,d$ must be greater than $c\,g\,h$ and, as $e\,g\,h$ must be a straight line, the point g does not lie on X X but to that side of it on which the tensile stress occurs.

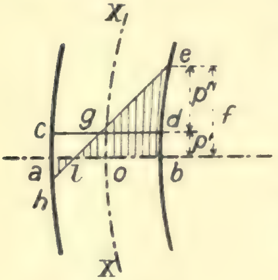


FIG. 50.

Under these conditions it becomes very difficult to construct equations giving the amounts and distribution of the stresses, and quite impossible to construct any workable formulæ for the strength of struts, with ratios of $\frac{L}{r}$ higher than that at which tension first appears.

The intensity of the maximum tensile stress must always be less than that of the maximum compressive stress, and as in wrought iron and steel the ultimate strength is greater in tension than in compression, such struts will always fail by crushing or buckling on the compression side first . (148)

In cast iron the ultimate strength in tension is only about one-fifth or one-sixth of the compressive strength, and in cast iron columns with high ratios of $\frac{L}{r}$ the commencement of failure is very often on the tension side. In such columns the ultimate strength, after tension first appears, becomes rapidly reduced with an increase in $\frac{L}{r}$ and it is the same when any material is used which has a greater ultimate strength in compression than in tension . . (149)

If cast iron columns having a large ratio $\frac{L}{r}$ were being designed it would be necessary to make allowance for this, but such cases are exceptional.

The whole of this chapter refers to struts in which the following conditions obtain :—

1. The dimensions of the section are the same throughout the whole of the length. If the strut should be of larger dimension at the centre, and taper towards the ends, the deflection curve will be quite different, and consequently the stresses will also differ with the degree of taper, and with the form of the lines of the tapering portions. Even a tapering of comparatively short portions at the ends will affect the results.

2. The sectional area is the same throughout. Even though the general outline dimensions of the section may be preserved from end to end, if the thicknesses of plates are reduced, or they are not carried up to the ends, or if in any way the area is reduced, the results will be less favourable than given by these equations.

3. These equations do not necessarily apply to struts formed of separate booms connected by open type bracing. That type is considered in Chapter X. If the bracing is of a type which divides the booms into short lengths, between the bracing connections, so that the curve of stress intensities is approximately continuous, the equations may be used, but unless these lengths are very short relatively to the general dimensions of the strut, the stresses will be greater at certain points than if there were a continuous web, the deflection will be increased, and consequently the ultimate load will be decreased.

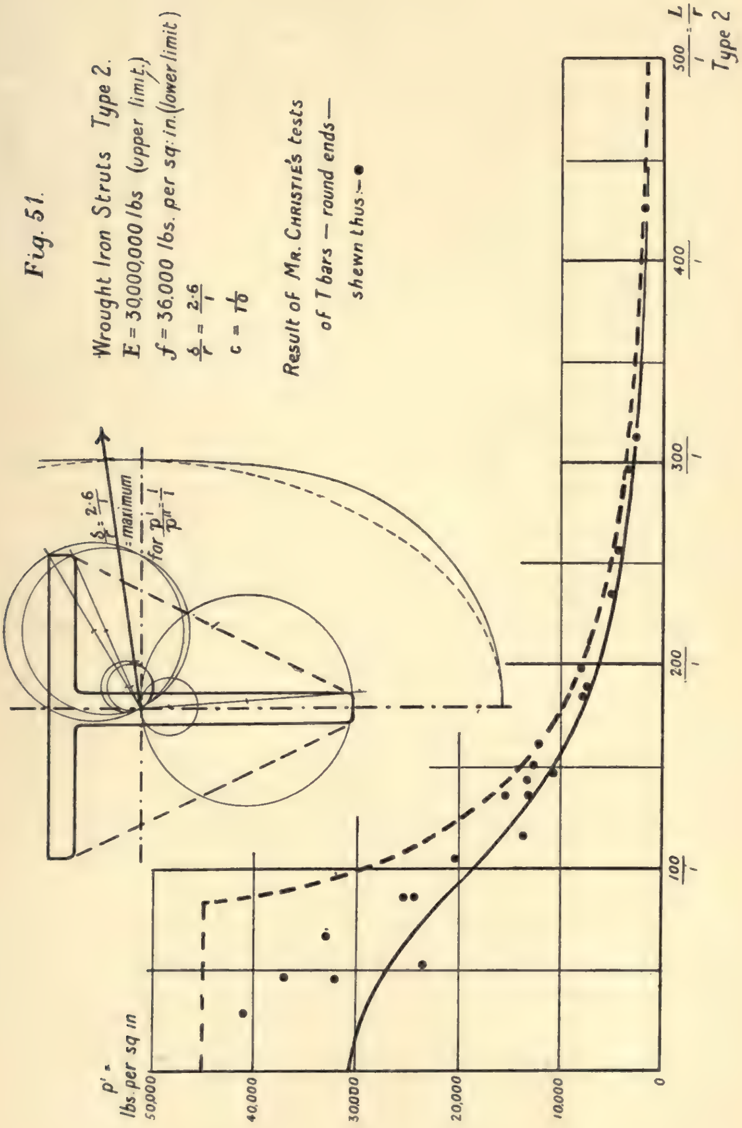
It is interesting and instructive to compare the results of various experiments with the theoretical predictions. Great numbers of experiments have been made, but in the majority of cases the results are of little real value owing to the conditions under which the tests were made being such as to prevent normal bending and deflection, or owing to insufficient observations having been recorded.

The experiments by Mr. Christie, to which more reference has probably been made than to any others, were made on angle, tee and other bars. The results are published in the Transactions of The American Society of Civil Engineers, Vol. XIII. Some tests were supposed to represent fixed end, Type 4, conditions but as the ends of the bars were insecurely fixed the results appeared little better than many with free ends and the results are valueless. Other bars had square ends and these, up to a certain point, should fairly represent Type 4 conditions, but only up to a very moderate limit of $\frac{L}{r}\left(\text{about } \frac{70}{1} \text{ or } \frac{80}{1}\right)$ above which tensile stresses should commence, and then, owing to the ends not being fixed, the results must be discordant. Other bars were supported on pins passing through their ends, but it was found that the friction on the surfaces bearing on the pins had a great effect on the results. Again, the pins were all set normal to the least radius of gyration, and the bars did not all deflect wholly in that direction. Thus the results of all these experiments cannot be expected to compare with any calculated results.

One set of Mr. Christie's tests was made on Tee bars mounted on spherical ends, bearing on flat plates in the testing machine. Although it was found that there was considerable friction between the balls and plates, the results are fairly representative of Type 2 struts. The results of these tests are shown in Fig. 51.

The Tee bars were of wrought iron said to have an average ultimate tensile strength of 49,000 lbs. per sq. inch. This, under the usual ratios, corresponds to a compressive strength of about 36,000 lbs. to 45,000 lbs. per sq. inch, and the

Fig. 51.


Wrought Iron Struts Type 2.
E = 30,000,000 lbs (upper limit.)
f = 36,000 lbs. per sq: in. (lower limit)
$\frac{\delta}{r} = \frac{2\cdot6}{1}$
$c = \frac{1}{16}$
Result of Mr. Christie's tests of T bars — round ends — shewn thus:— ●
$\frac{\delta}{r} = \frac{2\cdot6}{1}$ = maximum for $\frac{p'}{p''} = \frac{1}{1}$
p' = lbs. per sq in
50,000
40,000
30,000
20,000
10,000
0
$\frac{100}{1}$
$\frac{200}{1}$
$\frac{300}{1}$
$\frac{400}{1}$
$\frac{500}{1} = \frac{L}{r}$
Type 2

curve for ideal conditions is drawn corresponding to the upper limit of 45,000 lbs. and for the upper limit of E (in compression) of 30,000,000 lbs. In Fig. 51 is shown a section of one of the Tee bars and a determination of the approximate direction in which deflection has the greatest effect, this giving a ratio of $\frac{\delta}{r}$ about $\frac{2\cdot6}{1}$. The curve for practical conditions has been calculated for this value, for the lower limit of $f = 36{,}000$ lbs., and for $c = \frac{1}{10}$.

These two curves form the upper and lower limits between which it might be expected that nearly the whole of the results of tests would lie, and it will be seen how nearly this is the case. Out of 23 results, 3 lie below the lower limit, but 2, at least, of these differ so much from the mean results that it seems probable that there were some special adverse conditions. In the records of the tests there are results for four Tee bars of exceedingly small dimensions, being only 1 inch by 1 inch by $\frac{1}{8}$ inch, and as these were tested with spherical ends 1 inch in diameter, or as great as the diameter of the bars themselves, and as the radius of gyration is only about 0·2 inch, it could not be expected that really representative results would be obtained. These four results have not been included. One of them lies somewhat below the lower limit, two close below it, and one between the curves.

It will be seen that the majority of the results are grouped along the middle half of the space between the two curves.

In the Transactions of the American Society of Civil Engineers, Vol. XVIII., the results are published of a series of tests made by Strobel on columns built up of Z iron bars connected by lattice bracing. The section is shown in Fig. 52, and the approximate direction is shown for greatest effect of deflection due to an initial deviation of P In the diagram adjoining are shown the directions of the observed deflections which occurred in the 15 specimens. Fortunately, in this case, both the amounts and directions of deflection were accu-

Fig. 53.

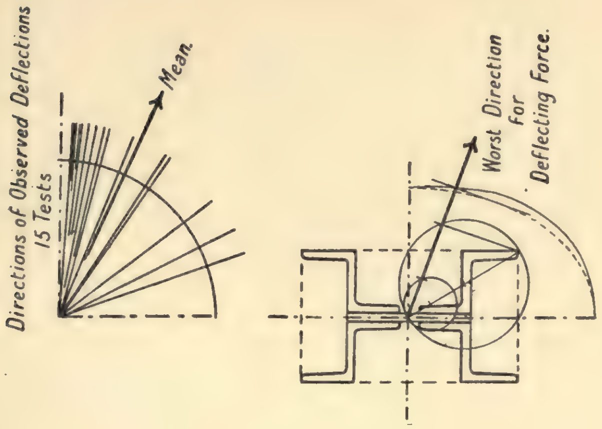
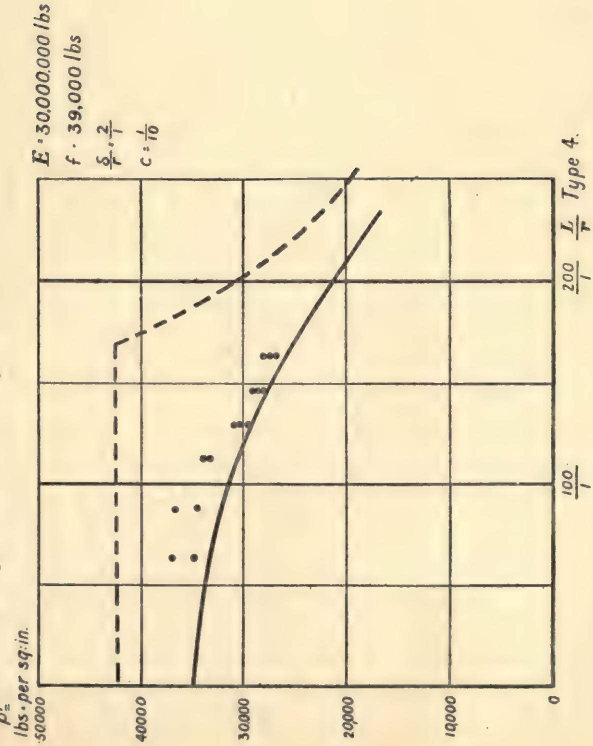


Fig: 52.

Results of Mr. Strobel's tests shewn thus:— ●

Wrought Iron Struts—Type 4.



rately observed and recorded. The mean of these directions is shown for comparison with the direction of greatest effect of an initial deflecting force.

These struts were tested with square ends and, as the proportions of $\frac{L}{r}$ were moderate, the question of tensile stress does not come in, and the results should represent conditions practically equivalent to Type 4.

The struts were made of wrought iron having an ultimate tensile strength of 49,000 lbs. to 52,900 lbs. per sq. inch, in tension, equivalent to about 39,000 lbs. to 42,500 lbs. in compression. The upper limit curve in the diagram is drawn for $E = 30{,}000{,}000$ lbs. and the lower limit curve for $f = 39{,}000$ lbs., $c = \frac{1}{10}$, and $\frac{\delta}{r} = \frac{2}{1}$, which is the ratio obtained from the section, for the direction of deflection indicated.

It will be noticed that all the results lie near to the lower limit curve. Owing to the outwardly turned wings of the Z bars being far from the axis and being thin in proportion to their projection, the defect pointed out in Fig. 9 exists. The projecting wings, also, measuring 2·5 inches and being only 0·3125 inch in thickness, have a ratio of projection to thickness of 8 to 1, which is therefore up to the limit stated in (154). In the published records referred to there are photographs of the struts after being tested, and these appear to show slight buckling near the centre of the length of some of the struts, which would, of course, lower the ultimate strength. The photographs also indicate that the permanent bending was greater at the centre than at the ends, showing that the conditions were not quite up to Type 4 and the results therefore lower.

It is stated that one of the advantages of this design is that it is economical in relation to quantity of web bracings because these are short, owing to the nearer wings of the Z bars being turned inwards. This could be, at the best, a very slight recommendation, as the bracings required are proportionately so light that any saving is overbalanced by the loss sustained in bringing these wings, forming part of the main section, so much nearer to the axis. Again, the bracing bars, meeting at their connections to the inwardly turned wings, do not intersect on the axis of the boom formed by the pair of adjacent bars, but at a considerable distance from it, and secondary stresses are introduced. On the whole the design is neither an ideal nor a very economical one. A section formed of four channel bars of the same total area and placed with their flanges pointing inwards would make a much better section.

Another series of tests, by Dragon, is reported in the Transactions of the American Society of Civil Engineers, Vol. XX. The columns were of large size, the lengths ranging from 16 feet to 25 feet. The sections were in the form of double channels connected by bracing bars, after the manner of Fig. 24, but instead of the channels being rolled sections they were built up of plates and angles. Two different sections were tested. In one the projecting wings of the angles were 2 in. × 0·25 in. giving a ratio of $\frac{8}{1}$, and in the other, $2\frac{7}{16}$ in. × $\frac{5}{16}$ in. giving a ratio of $\frac{7 \cdot 8}{1}$. These ratios are up to the limit stated in (154). Photographic views are given of these columns after failure, and in every case failure took place by local buckling of these angle wings near the centre of the strut.

Many other series of tests are on record, but it is difficult to find any under conditions quite suitable for giving reliable data. In making tests to procure data or to confirm or test theories, special attention is necessary with regard to the type of end supports. As it is impossible to be sure that fixed or flat ends are properly secured or adjusted relatively to the axis, it appears that no reliable data can be expected except from tests under Type 2 conditions, with ends free. This can best be done by fitting spherical ends to the strut, taking care to see that the centre of the sphere lies on the axis of the strut. The length should be measured between the centres of the spheres. If these spheres were of hard material and arranged to bear on hard flat plates in the testing machine, the amount of frictional resistance would be as small as practicable, and results would be valuable. In most cases tests have to be made with the struts lying horizontally. It is then arranged for the weight of the strut to be counterbalanced to avoid deflection under its own weight. This is often done at the centre only, but when the ratio of $\frac{L}{r}$ is great it should be done at several points.

Tests of struts with pin ends never give satisfactory results. In addition to disturbances caused by friction between the bearing surfaces and the pins there is the fact that the strut is free to deflect in a plane transversely to the pin axis, but is restrained in the direction of that axis. The conditions are therefore different, being nearly of Type 2 in one direction and approaching Type 4 in the other, while for intermediate directions the conditions are much more complicated. Such struts seldom deflect exactly transversely to the pin axis, and it is impossible to separate the data obtained into the components due to the varying conditions and directions.

Results of tests under Type 2 conditions are the only ones which are of any value for assistance in designing, and when such results are available they are readily convertible to suit any other types of conditions.

CHAPTER VIII.

SOME PRACTICAL POINTS AFFECTING DESIGN.

BEFORE the actual design of a column or strut is commenced all the preliminary information must be to hand, such as the quality and strength of the material, the permissible stress intensities, the nature and amount of the load to be carried, the length, the nature of the supports or end connections, the working position of the strut, whether vertical, horizontal, or inclined, and particulars of all conditions likely to affect the principal stresses, such as wind pressure, etc. It is nearly always the case, that the load and length are both given, and the section has to be designed accordingly, but sometimes it is required to find the greatest length consistent with safety for a given section and load.

END CONNECTIONS OR SUPPORTS.

The type of the strut has to be determined. In Chapter VII. various types have been already referred to, and it was explained that the Type Number represents the complete, or fractional, number of semi-curves of the elastic type, which make up the curve of bending. By a semi-curve is meant that portion which lies between a point of contrary flexure and the first point along the curve where the axis becomes parallel to the line of action of the external force, or load P.

Type 1.

Type 1 applies to the extreme case of a column securely fixed to a foundation, or attached at one end so that the axis at that end cannot deflect angularly, but with the other end entirely free and without support or restraint of any kind. Such conditions are not very common, but are sometimes met with. For example, at many railway stations there are water tanks for supplying locomotives. Each of these tanks is placed on the top of a single column, generally of cast iron and this is an undoubted case of Type 1.

In some towns, shelter roofs may be seen over market places, etc. These roofs are often carried by a series of columns, and no bracings are provided between any of them to keep them upright. The roof trusses may be secured to the tops of the columns, but such attachments are too slight, in proportion to the sizes of the columns, to have much effect in preventing angular movement or deflection of the axis. Of course, the tops of all the columns are tied together, and one could not move unless all the others moved with it, but if they are all fairly equally loaded there is nothing to prevent them doing so. This is practically a case of Type 1.

The conditions of Type 1 are the worst possible, unless the foundations could yield and allow the axis at the base to deflect angularly; but in all such cases the foundations have to be heavy enough to ensure stability against wind pressure, etc., and are therefore rigid enough in practice.

Types Intermediate between 1 *and* 2.

No conditions of support, or end connections, can ordinarily exist such as would produce a type anywhere between Nos. 1 and 2.

Type 2.

Struts of Type 2 are indiscriminately spoken of as having free ends, round ends, pin ends, hinged ends, etc. It is impossible, in practice, to produce the exact conditions of Type 2. The load, or external force, would have to be applied at a single point on the end; but this could not be done, as the material of the strut would be overstrained locally at that point. A load cannot be, in any practical way, spread over any area on the end of the strut without, to some extent, impeding its freedom to deflect. Even when, in making laboratory tests, struts have been fitted with spherical ends bearing on flat plates, it has been found that the friction between the ball and the plate impedes free movement, and a ball of larger radius does so more than a smaller one. At the same time such friction is uncertain, and cannot be depended upon, and no mere frictional support should ever be credited in settling the type.

Frictional resistance to deflection is, in practice, often met with—for example, when the ends of a strut are secured by pins passing transversely through them. The friction between the pins and the surfaces on which they bear prevents perfectly free deflection ; but in testing such types it is often found that when a certain load is reached the friction is overcome and the bearing surfaces slip on the pins, allowing the deflection to increase suddenly, and sometimes the momentum of the parts thus moving will carry them beyond their position of free equilibrium, and when they come to rest the friction will hold them there, and the strut will remain deflected more than its natural amount. That friction does not assist the strut.

Struts with pinned ends are to a great extent constrained to deflect in a direction at right angles to the pin axis, but it must not be assumed that they will always do so. An examination of the records of column tests will show that in a very large proportion of the tests deflection took place in a direction other than at right angles to the axis of the pin. The direction of the deflection depends partly upon the form of the section, and partly upon any initial deviation of P from the axis of the strut and the direction of such deviation. These possibilities must be taken into consideration.

Type 2 is the most unfavourable condition of support for a strut attached at both ends to other parts of a structure. There may be auxiliary conditions causing greater deflection than would occur in a perfectly free-ended strut, but such are not to be considered as part of the direct strut problem, but as subsidiary, and must be investigated separately.

In Type 2, as we have seen, the curve of bending consists of two portions of the elastic curve of Type 1, placed end to end, and therefore, for a given section and a given load, a strut of Type 2 can be made twice as long as if of Type 1.

Type 4.

The most favourable conditions possible for support of a strut are, when both ends are fixed in such a way that the longitudinal axis cannot deflect at the ends but must remain in a fixed direction there, however much it may deflect between these points. In practice it is impossible to realise this condition. No matter how rigid the foundations or parts of the structure may be to which the ends are secured, they have some elasticity, and will themselves deflect when the reactions of the column stresses come upon them.

If a column is bedded on a massive foundation, and the upper end secured to some framework or part of a structure much more rigid than itself, it may be taken as being of Type 4. Comparatively slight irregularities will, however, act prejudicially. For example, if a column carrying a heavy load was placed eccentrically on a foundation block, it might have a very appreciable effect. As in Fig. 54, the downward reaction of the column does not pass through the axis of the foundation block or through the centre of the bearing of the block on the sub-soil. The distribution of pressure over a horizontal section or over the base of the block will be as in the small diagram below. The material of the block, being elastic, will be compressed more on the side where the pressure is greatest, and the subsoil will yield more on that side, resulting in an angular deflection which will be transmitted into the column, as shown by dotted lines. The extent of such movement is purposely much exaggerated in the figure, but however slight it may be it acts disadvantageously. Columns carrying heavy loads should always be provided with foundations symmetrically arranged about the vertical axis, even if something else has to give way to them.

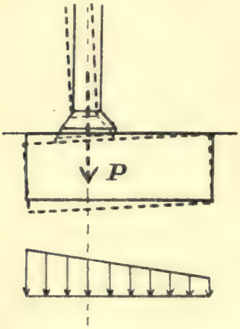


FIG. 54.

It may be here pointed out that under no circumstances can the maximum intensity of stress or pressure on a bearing surface be reduced by adding to it on one side only, if it is originally symmetrical. As an illustration, let Fig. 55 represent a column base resting on a foundation block, and carrying a load, P Let P be 20 tons, and the column base be 2 feet square. The bearing area is 4 square feet, and being symmetrical about the line of P the intensity of pressure p will be uniform and equal to $\frac{P}{4}$ = 5 tons per square foot, as in the upper small stress diagram. Now suppose that the column base is extended in one direction, as indicated by dotted lines, to the extent of 6 inches. The base is now 2·5 feet × 2 feet = 5 sq. feet, and the mean intensity of pressure is $\frac{20}{5}$ = 4 tons per sq. foot. The load P is, however, now 0·25 foot from the axis of the bearing surface, and the moment of the eccentric action is 20 × ·25 = 5 foot-tons. The corresponding stresses on the surface will be distributed over it in the same way as those due to an eccentric load over a section of a column. Therefore on one side of the axis of the area there will be a downward pressure, and on the other side a lifting or tensile tendency. The maximum intensity of such pressure will be given by

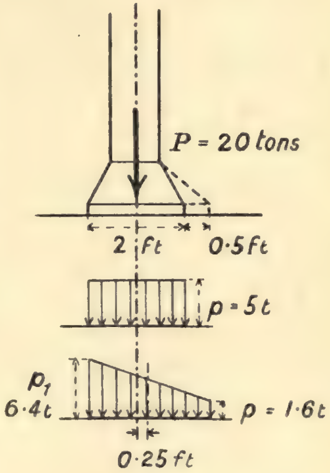


FIG. 55.

$$\frac{\text{moment} \times \text{distance from axis to edge of area}}{\text{moment of inertia of surface about axis}}$$

and the numerical value is

$$\frac{5 \times \frac{2{\cdot}5}{2}}{\frac{2 \times 2{\cdot}5^3}{12}} = \pm\ 2{\cdot}4 \text{ tons per sq. ft.}$$

Combining these with the direct pressure, we have the resulting maximum and minimum intensities

$$4 + 2{\cdot}4 = 6{\cdot}4 \text{ tons per sq. ft.}$$

and

$$4 - 2{\cdot}4 = 1{\cdot}6 \text{ tons per sq. ft.}$$

at the extreme edges, as indicated by the lower small stress diagram. Therefore, although the average intensity over the surface has been reduced from 5 to 4 tons per square foot, the maximum intensity has been increased from 5 to 6·4 tons per square foot.

It will be seen that a comparatively small eccentricity makes a considerable difference in the intensity, and such eccentricities should be carefully avoided, unless there are some very special circumstances calling for them. If the column base in the example had been originally made eccentric, the maximum intensity of pressure on its base would be reduced by cutting away 6 inches on the projecting side. The principle of the foregoing applies to all cases of pressure on a surface and to stresses on a section.

Types Intermediate between 2 *and* 4.

Between Types Nos. 2 and 4 there may be any intermediate conditions of support.

If the strut forms part of a braced structure or girder, and is riveted to the other members which meet at its ends, the axis is, to some extent, restrained from deflecting. If the members to which it is connected are heavy in proportion to the strut itself, the support will be considerable, and if these heavy members are not liable to deflect under their own stresses, such a strut will be nearly of Type 4. On the other hand, if the heavy members are struts carrying stresses as great in proportion as in the strut under consideration, these members will deflect under their own stresses, and the value of the end connections may become nil, and the type of the strut become practically No. 2.

It is never the case that a strut can be connected to other members which are absolutely unyielding, and very rarely to members which are otherwise unstressed. There is, therefore, a practical limit in the average structure to the value of such connections. Fidler ("Practical Treatise on Bridge Construction") has arrived at the conclusion that no connection can be of a greater value than to give the strut a strength equal to that of a free-ended strut (Type 2) of $\frac{6}{10}$ of its length. This appears to be fairly reasonable, although, under very favourable circumstances, somewhat on the low side. The ratio $\frac{6}{10}$ corresponds to a Type $\frac{2 \times 10}{6} = 3 \cdot 33$, and we may therefore say that in no case should a member be credited with a Type No. greater than $3 \cdot 5$.

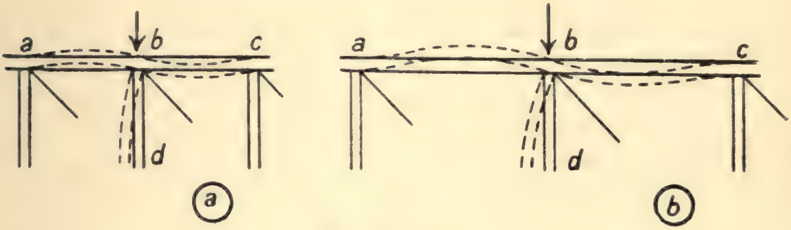


FIG. 56.

The value of the support given by connected members depends not only on their relative strengths or weights, but also upon their lengths and dispositions. For example, in Fig. 56 (*a*) a strut *b d* carrying a certain load might be connected to a continuous member *a b c*, and the tendency of *b d* to deflect would also cause *a b c* to deflect (as indicated by dotted lines), the amount of the deflection depending on the relative stiffnesses and lengths. Then, again, if as in Fig. 56 (*b*) *b d* was a strut of the same dimensions as before and carrying the same load, and if *a b c* was a continuous member of the same section as before, but if the lengths *a b* and *b c* were greater, *a b c* would clearly offer less resistance to deflection and give correspondingly less assistance to *b d* In such cases no general rules can be laid down for determining the equivalent Type No., and it is generally necessary for the designer to consider the circumstances attending each problem and determine the type.

If a row of columns carries a line of continuous girders, as in Fig. 57, being rigidly connected to them and having well-spread bases attached to solid foundations, they will be very nearly in the condition of Type 4, as regards bending in the direction of the girders, if the girders carry a dead load only and are not subjected to varying deflections. Such girders are generally very stiff compared with the columns, and will practically prevent angular deflection of the axes at the upper ends of the columns. If, however, the girders carry a moving load, such as a crane, any girder will deflect when the load is over it (as at *a*), and as the girders are stiff, they will bend the column as indicated by the dotted lines. At first sight it might appear that such bending would very much reduce the load-carrying capacity of the column, but, as the girders are stiff, the angular movement over the column will be slight, and, as a rule, insufficient to produce any large bending stresses in the column. The direct load on the column will however not act along the axis, as there is an initial deflection. This will bend the column still more, but it will be seen that the angular deflection of the axis at the upper end cannot increase unless the girders are also bent to a greater extent, and they still assist in stiffening the column, although not to such an extent as if they were not themselves deflected by their load. If the angular deflection of the end of the girder

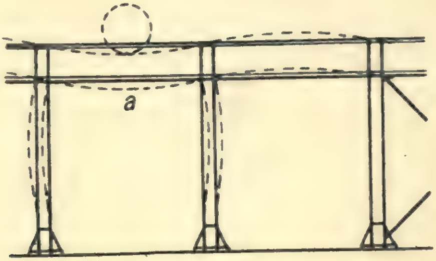


FIG. 57.

is less than the natural angular deflection of the column axis would be if the end were free, the conditions are better than those of a free end. An attachment of a column to a girder should not be credited with a value greater than that of attachments between members in a braced structure.

It often happens that a strut is fairly well supported in one plane but not in other directions, and it then depends entirely on the local circumstances whether it should be made stiffer in the unsupported direction only, or made stiffer generally. If there is any doubt as to the direction in which the conditions are least favourable, it is always better to make trial calculations to find out.

Another point which arises in connection with struts of a higher type than No. 2 is the effect of initial deviation of the axis. Take a strut of Type 4, Fig. 58, having its axis X X to the left of the geometrical centre. The assumption of Type 4 requires that the direction of the axis at P and P_1 remains constant, and the problem does not depend upon relative positions of forces and axes, because P does not act alone but in conjunction with a couple, consisting of the forces required to maintain the direction of the axis. The result is that the two ends of the strut are being forced toward each other, and their direction maintained irrespective of what efforts they may exert to deflect. No definite positions can therefore be directly assigned to the components of these forces, and the problem becomes entirely one of the elasticity of the material and its distribution. The strut in Fig. 58 (*a*) would deflect as in (*b*), and as the material is non-homogeneous, or not symmetrically disposed, the points of contrary flexure in the curve of bending may not be located at one-fourth of the length from each end, as they would be under ideal conditions. These points may be

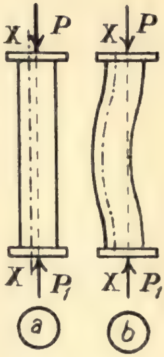


FIG. 58.

nearer the ends or the centre, but in whatever direction they are displaced the effect will be similar, namely, to increase the curvature at some points and reduce it at others.

Wherever the points of contrary flexure are, the portion of the strut between them is in the condition of Type 2, having a length equal to the distance between these points, and with conditions of loading of an ideal nature. This must be so, because at the points of contrary flexure there can be no bending moments, and the strut might be cut through transversely at these points without altering the curvature, and as there are no bending moments, the pressure on these surfaces must be uniform, and the centres of the pressures must pass through the axes of the sections. In addition to the pressures on the surfaces there would be shearing forces, which, compounded with the direct pressures, would have a single resultant for each surface, and therefore, as the portion of the strut is in equilibrium, these two resultant forces must act in the same straight line. Therefore, the portion of the strut between the points of contrary flexure is under ideal conditions even if the real axis does not coincide with the geometrical axis, provided that they are parallel to each other.

The two end portions of the strut will be in a like condition, each portion corresponding to a strut of Type 1.

If the axis is irregular, these conditions will be correspondingly disturbed. It will be readily seen that the net result of such departures and irregularities is, that between two points at a quarter length from each end, which are the points of contrary flexure for ideal conditions, that part of the strut will be in a better or worse condition than a strut of Type 2 according as the points of contrary flexure move nearer the centre or the ends. If they move toward the ends, the centre half-length of the strut will be exactly in the state of a strut of Type 2, in which the load does not act along the axis. On the other hand, if the points of contrary flexure move towards the centre, each end quarter-length will be exactly in the state of a strut of Type 1 with the load not acting along the axis. In both cases the effect on the strut as a whole is the same, and the strength will be the same as one of Type 1 of one-quarter the length, or as one of Type 2 of one-half the length, the loading conditions not being ideal.

The general maximum departure from ideal conditions may be taken to be the same as in the case of Types 1 and 2, and therefore the same equations (135) and (145), etc., will apply to Type 4, or to any type intermediate between 1 and 4.

Workmanship.

In manufacturing struts, and fixing them in place in structures, it is necessary to see that they are straight within the prescribed limits. The importance of this will be realised on comparing the results given by equations for ideal conditions with those given for practical conditions. There are many things which will cause irregularities in the alignment of a strut. Imperfect straightening of rolled bars, drifting parts into place when rivet holes are out of register, irregularities in the length of bracing bars, riveting along one side in advance of another, and many minor causes, will throw the axis sometimes considerably out of straight. In good work such effects must be guarded against. Once such defects have been allowed to arise, it is impossible to satisfactorily cure them without doing the whole of the work over again. Cold straightening, chipping, or fitting, may apparently remove such a defect, but will leave internal stresses and other faults, which may be as bad, or worse. In the case of important struts too much care cannot be taken, as any initial defects may increase after the strut is loaded. In less important work, or where labour conditions, etc., are not of the best, greater allowance should be made for initial irregularities, by increasing the factor c in the equations.

Materials of Construction.

In modern structures the material chiefly used is what is generally termed mild steel, that is, steel containing a moderate percentage of carbon, and having an ultimate tensile strength ranging from about 26 to 32 tons, or 58,000 to 72,000 lbs., per square inch. Cast iron is often employed for columns of moderate size, and particularly when carrying steady loads. Timber is much used in structures of a temporary character, but not to a great extent in permanent structures.

The equations given herein apply to struts of any material for which the values of E and the ultimate strengths in tension and compression are known. It is also advisable to know the limit of elasticity, so as to ensure that the maximum working stress intensity decided upon is not too near that limit.

The following table gives the values of the various factors for materials of the kinds generally used in constructional work.

—	—	Tension		Compression	
Material	—	Ultimate Strength	Modulus of Elasticity	Ultimate Strength	Modulus of Elasticity
		lbs. per sq. in.	lbs.	lbs. per sq. in.	lbs.
Wrought Iron	from	45,000	27,000,000	36,000	26,000,000
	to	55,000	31,000,000	45,000	30,000,000
Mild Steel .	from	62,000	29,000,000	50,000	28,000,000
	to	70,000	32,000,000	60,000	31,000,000
Harder Steel .	from	85,000	29,000,000	70,000	28,000,000
	to	110,000	32,000,000	90,000	31,000,000
Cast Iron . .	from	14,000	12,000,000	70,000	12,000,000
	to	26,000	15,000,000	110,000	15,000,000

It must be remembered that these are only general values, and, before designing any important strut, it is necessary to know with some exactness the nature of the material to be used, and to specify the limits of strength, etc., when ordering;

and also to see that, when manufactured, proper tests are made to ensure that the quality is as specified.

In some cases, steel of a greater strength than above is used. The ultimate strength may be raised to about 120,000 lbs. per square inch, or more. Again, steel alloyed with one or more of the rarer metals, such as nickel, vanadium, chromium, etc., may be employed. In all such cases the necessary data must be carefully obtained.

The most important factor is the Modulus of Elasticity, E, for compressive stress. If, for any particular material, it is known that E is liable to vary more than in the case of ordinary mild steel, as shown in the table above, it will be necessary to proportionately increase the factor c in equation (145). In all cases the upper limit of E should be used in the equations, as the constant provides for the variation, and therefore for conditions generally worse than if E was quite uniform and at its lower limit.

Factors of Safety.

In any case, such as that of a tension member, where the stress intensity practically varies directly as the load, it is immaterial whether the crippling load is calculated and divided by the so-called factor of safety to determine the working load, or whether the working stress intensity is taken as the same fraction of the ultimate strength and the load required to produce the working intensity called the working load. The same result will be obtained in both instances.

In a strut, if the conditions are taken as ideal, the stress intensity will vary directly as the load up to the point where bending commences, but after that the intensity increases with great rapidity. With a high ratio of $\frac{L}{r}$ the maximum stress intensity at the instant of bending is a small fraction of the ultimate intensity, and the ratio of the two intensities may be less than the value of the required factor of safety. If the practical conditions are taken into account, the rate of increase of the stress intensity is modified, but is still very far from being uniform with a uniformly increasing load.

The only satisfactory and practical proceeding is to calculate the crippling load, and divide it by the factor of safety for the working load. The reverse process must, of course, be employed when designing a section; that is, multiply the given working load by the factor of safety, and use the result in the calculations as being the required crippling load.

The value of the factor of safety required varies with the nature of the load on the strut, the material, and many other factors, and each case must be considered separately.

It is sometimes made a practice to allow a higher factor of safety in struts than in tension members. When using the equations given herein it is not necessary to do so, unless there are unusual circumstances or contingencies to provide for. In making allowance, in the equations, for an initial deviation of the load from the axis, one of the contingencies is thus already provided for, and one which also occurs in tension members, but is then covered by the ordinary factor of safety. The same remarks apply to the allowance made for a variation in E If, therefore, in designing a structure, the factor of safety used as above is given the same numerical value for both tension and compression members, the latter will in reality be slightly less stressed in proportion.

Again, in a strut, unless the ratio $\frac{L}{r}$ is small, the maximum stress intensity under the working load is lower than in a tension member, and it is not necessary to make as great allowance for fatigue, or for dynamical action of rapidly applied loads.

In designing braced struts, the sizes of the bracing bars must be determined in the same way as the main section, using the working load on the strut, multiplied by the factor of safety, as the crippling load to be provided for. The equations for stresses in bracing bars must not be used in any other way, as they only give approximately correct values for the ultimate loads and stresses. For less loads or stresses they will not give results with anything approaching accuracy.

The principle of using the factor of safety as a ratio between the ultimate load and the working load is the only rational one. Otherwise, in a structure, it would probably occur that one member would be stressed up to its ultimate limit under a less overload than the others; and under such load the structure would collapse just as certainly as if these other members were made lighter so as to reach their respective ultimate stress limits all at the same time, or under the same load, and all such difference in material would be direct waste, or even worse.

Joints in Members.

Much difference of opinion has existed as to the necessity, or otherwise, for providing cover-plates, rivets, etc., at a joint in any compression member up to the full strength of that member. If the meeting ends could be made, by machining or otherwise, to butt against each other uniformly over the whole area, it would only be necessary to provide some means of preventing lateral movement or sliding over each other. On the other hand, if the ends are not machined, it is obviously necessary to make joints in the same way as for a tension member.

In practice, it is quite impossible to machine joints with sufficient accuracy to avoid all necessity for cover-plates, etc., and everything depends upon the class of work whether riveting should be provided up to half, two-thirds, or more, of the whole strength. In any case, it is inadvisable to provide less than one-half, and the really safe course is to provide for the whole.

When a strut is continuous through a number of points of lateral support, it is best to arrange joints as near as possible to points of contrary flexure, if such can be approximately located, as there will then be little more than the amount of the direct stress on the section, and consequently less necessity for reinforcement.

CHAPTER IX.

LOCAL AND AUXILIARY FLEXURE.

In a braced strut the portions of the booms between the points of attachment of the web bracings, such as *a b*, *b c*, *d e*, in Fig. 59 (*a*), have to carry the main compressive stresses without intermediate support between these points, and each of them is therefore in itself a short strut. The booms are invariably continuous, but that does not assist in stiffening the various parts, as they can deflect alternately in opposite directions as indicated in (*b*), when each of the parts subtending one bay of the bracing is in the condition of a strut of Type 2. The bracing bars, being connected to the booms, must to some extent tend to prevent deflection, but these bars are generally so light compared with the booms that their assistance is negligible, especially as they are themselves stressed each alternate one in compression, and have a tendency to deflect. As, under such deflection of the booms, there will be a point of contrary flexure, practically, between each bay, and therefore no bending moments there, each of the short portions may safely be treated as under ideal conditions of loading.

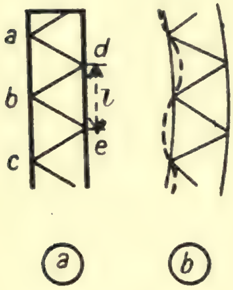


Fig. 59.

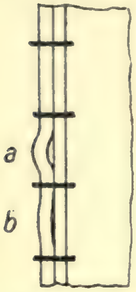
In designing a braced strut, it is necessary to determine the total load-carrying capacity of the unsupported lengths of the booms, and limit the total stress in each of them due to the direct action of the load, combined with the stress due to bending of the strut as a whole, to that amount. It is not sufficient, as is frequently done, to design the section to suit the direct and main bending stresses only, and then see that the ratio $\frac{l}{r}$ for each bay of the booms is as small as for the whole strut. Under such a procedure the mean stress intensity in the boom, on the concave side when bent, will be up to the assigned limit, and then the local deflection, or bending, in each bay will increase the intensity at one edge of the boom, and the maximum stress intensity will there be above the limit. With high ratios of $\frac{l}{r}$ such increase would be very great, and might be dangerous. The thrust-resisting capacity of the individual portions of the booms should first be calculated for the limiting value of f and the result (say P_1) divided by the area of the boom will give a value f_1 which should then be substituted for f in the equations for calculating the resistance to the main load, P As we may allow for ideal conditions of loading, we take equation (135), and write f_1 for p'

whence $$f_1 = \frac{E\,\pi^2}{\left(\frac{l}{r}\right)^2}, \quad . \quad . \quad (150)$$

the constant for Type 2 being used.

With a high ratio for the short portions of the boom, it will be found that f_1 is considerably less than f As a general rule, it would be very wasteful to design the whole strut on such a reduced value, and it will nearly always be more economical to reduce the ratio $\frac{l}{r}$ for each bay, by shortening the distances between the bracing attachments, that is, to shorten the bays. This, of course, means more material in the bracings, but increasing the booms would require a much greater additional quantity.

When the various parts of a built-up strut are riveted together, it is necessary to ensure that the rivets are pitched closely enough together. If the length of a portion of a bar or plate between two rivets is too great, it may buckle as indicated at *a*, Fig. 60, when stressed. The curve of the bent part would be similar to that of a strut of Type 4, but as plates when riveted to other members are drawn closer to them at the rivets than between, there is nearly always a slight bulging (as at *b*) even before being stressed, and it is not advisable to count upon anything like the maximum stiffness under Type 4 conditions. The rivets should be considerably closer together than the purely theoretical distance, and the pitch of the rivets should never, under ordinary circumstances, be greater than about ten or twelve times the thickness of the plate or bar. This refers to parts stressed up to the maximum stress intensity . (151)



.Fig. 60.

Any plate which extends, in a transverse direction, to a considerable distance, compared with its thickness, without support is liable to buckle under stress. No rules can be given for determining the result of varying the thickness, but experiments show that no such plate, in iron or steel, should extend, unsupported, more than about thirty times its thickness. . . . . (152)

A curved plate, owing to its shape, is much stiffer than a flat one, and requires less assistance from other members. Here, again, experiments indicate that the thickness of any curved plate, of iron or steel, supported along both edges, or of complete circular form, should not be less than one-thirtieth of the radius of curvature. Thus, the thickness of metal in a tubular strut should not be less than $\frac{1}{60}$ of the diameter. There is little doubt that this applies to tubes of moderate dimensions, but there are no means of experimentally determining the limiting dimensions. Tubular struts of large sizes are always built up of sectional plates, jointed and stiffened. In such cases it is not convenient to make plates of a thickness as great as $\frac{1}{30}$ of the radius, and when thinner they are in a state intermediate between that of the plain tube and the flat plate. . . . . (153)

A flat plate or rib projecting from the main section, and unstiffened along its outer edge, is in a worse condition than a flat plate supported along both edges. Such a rib is equivalent to about one-quarter of the width of the flat plate, and therefore no such rib, in iron or steel, should project more than about eight times its thickness . . . (154)

If the projecting plate is curved, it is stiffer than if flat. Probably the best investigation of matters connected with the local flexure of such plates, etc., is recorded by Lilly in his work on "Plate Girders and Struts," and in other writings referred to therein. This part of the subject requires still closer investigation, and it seems probable that the main equations for struts may be developed and extended so as to cover many phenomena of local flexure.

Bases, Caps, etc.

When a short metal bar is compressed in a testing machine it shortens in length, and at the same time expands laterally. If the bar has flat ends bearing on the blocks of the machine, the expansion is found to be not uniform, but greater near the centre than at the ends, as indicated in Fig. 61 (*a*). When a tube is tested in the same way it generally fails, as shown in (*c*), by bulging out near each end. The cause of the expansion of the bar being less at the ends is the friction between the bar ends and the bearing blocks, which prevent the metal spreading so freely as it does near the middle of the bar. When the tube is first subjected to load it expands, but is prevented from doing so at the ends, as above, and is then in the condition shown in (*b*). It will be seen that at the ends, where the tapering occurs, there is an inclined thrust on the metal, having an outward component all round the tube, and, owing to the curvature and distortion, the stress intensity will reach the elastic limit sooner at these points than nearer the middle, where the tube remains nearly parallel. The result is, that the tube fails by bulging out permanently, as in (*c*).

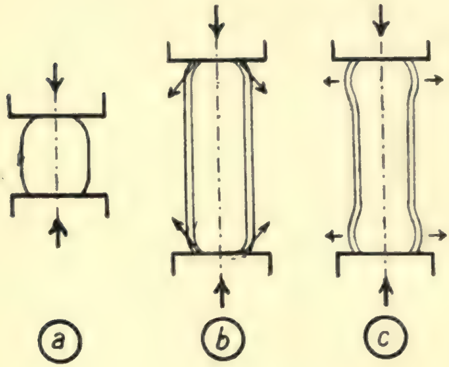


FIG. 61.

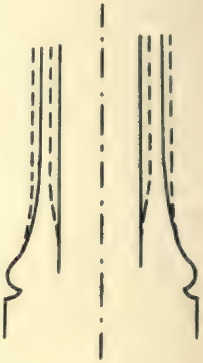


FIG. 62.

When a cylindrical or prismatic shaft of a column is mounted on a very solid base, and surmounted by a rigid cap, these conditions are set up. If the column is of cast metal, the outer diameter should be increased towards the cap and base, as indicated in Fig. 62, so that when the shaft expands the curves near the ends will run into the enlarged portions, as indicated by the dotted lines, without overhanging as in Fig. 61 (*b*).

If the column is built up, it is not so easy to provide for the above conditions, but by judicious reinforcement on the outside near the ends the effects can be minimized. If an end is formed as in Fig. 63, whether it is riveted, pinned or attached in any other manner, the plates marked *m*, and called tie plates or batten plates, prevent lateral expansion and act detrimentally. In the same way, as in the case of the diaphragm, and as in Fig. 71, the fault can be minimized by attaching stiffening plates outside, as at *n n* (Fig. 63). If these stiffening plates are tapered off as indicated, their effect will be improved, as the reinforcement will not take place so suddenly.

This question is not of so great importance at the ends of a strut of Type 2, because in that case there are no bending stresses at the end, and, the main stress intensity not being up to the limiting value, the spreading tendency is less. Type 2 conditions at an end support are, as already stated, never attained in practice, and there are always some bending stresses.

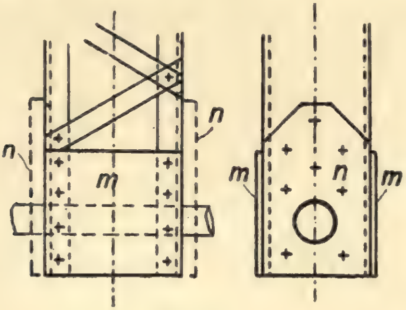


FIG. 63.

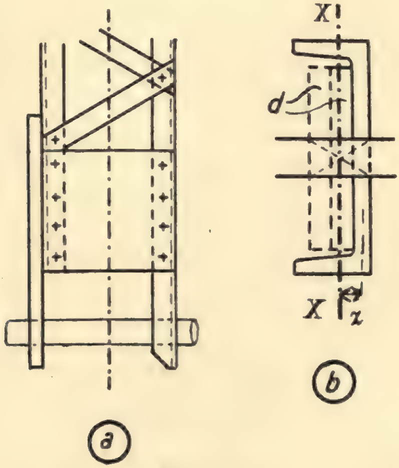


FIG. 64.

Sometimes, in structures, the ends of struts are formed as in Fig. 64. To allow other members to pass, or for other reasons, the bracings are terminated some distance from the end pin or connection. As at the right hand side of view (*a*), the main member may be extended to receive the bearing pin, or the extension may be, and frequently is, made by a flat plate as at the left side. This detail is objectionable, and should be avoided if possible. There being no bracings between the extensions, they have to carry any transverse component of stress, or shearing force, as cantilevers, and are then subjected to bending moments. Should there be no transverse force, the extensions are still liable to deflect, because they must act as struts practically of Type 1. These extension pieces require extra stiffening, and, if they are unavoidable, such stiffening should not be neglected. It may, of course, happen that if the main ratio $\frac{L}{r}$ of the strut is great, the stress intensities near the ends are very low, in which case an extension of a channel or other bar may already be stiff enough to resist local flexure.

In connection with pinned end connections, and especially if extensions, such as just referred to, exist, there is a matter which must be noted. Take, for example, a pin bearing in the web of a channel, as at the right of Fig. 64. A cross section is as shown in (*b*). The pin bears on the web of the channel, and the neutral axis of the channel is at X X The external force, or load, is then applied at the distance z from the axis, and the resulting bending moment is considerable. In such cases extra bearing plates should be added, so as to make the centre of the pin bearing coincide as nearly as possible with the axis of the channels. Care must be taken to provide enough rivets for attaching these extra plates, not only to transmit into the channels the pressures they receive from the pin, but also to ensure that the attachment is not too yielding.

If the strut members are not extended beyond the plates, but the joint formed with a pin, as in Fig. 63, there is no need for these reinforcing plates so far as getting the centre of pressure on to the axis is concerned, and if there is sufficient bearing surface for the pin on the web, or on the web and outer plate, they may be omitted. In such a case the tie plates serve to distribute the stresses, and transmit them directly into the longitudinal members of the strut. The tie plates are then useful, but they serve no other purpose. As part of the system of bracing in the strut web, tie plates are not only useless but detrimental. They are, however, so generally used that they have come to be thought necessary, and rules are sometimes given for their dimensions—such as that they should be equal to, or one and a half, or twice their width, but no reasons are ever given for the rules. It is no more necessary to provide tie plates for carrying shearing stresses at the end of a strut, than it is to plate the end bays of a braced girder instead of using diagonals.

Fig. 65 is an illustration of a riveted joint between a vertical strut in the web of a girder, a diagonal and the boom. The arrangement of this joint is probably as good as it could be made. The two channels forming the booms of the strut are prevented from spreading by the connecting, or tie, plates at their ends, but any tendency to the bulging effect of Fig. 61 on the webs of the channels is considerably counteracted by the two gusset plates forming an arrangement as suggested in Fig. 71.

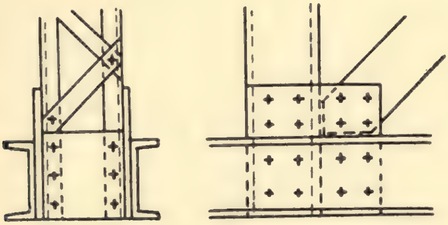


FIG. 65.

It frequently happens that sufficient care is not taken in arranging bracing bars so that their axes intersect on the axis of the boom. They then almost invariably intersect outside, as in Fig. 66. The resultant of the stresses in any pair of intersecting bars is always a single force parallel to the axis of the boom, and is indicated by p There is then a bending moment pz which has to be resisted by the local stiffness of the boom, assisted by the stiffness of the bracing bars if they are attached by more than one rivet each. Local deflections and stresses are the result, and in a strut of any importance these intersections should be carefully located. An example of this is described in Fig. 98.

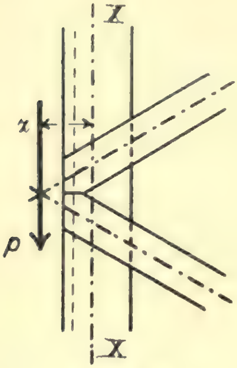


FIG. 66.

During comparatively recent years many struts have been constructed with independent booms, merely connected together at intervals by so-called batten plates, instead of by proper systems of bracing. Such a strut is indicated in Fig. 67, and, if of Type 2, will bend to an irregular shape, as shown, instead of into a regular curve. Owing to want of diagonal bracing, there must be severe local stresses unless the strut is very lightly loaded. In a strut in which, for any local reasons, there may be a great surplus of sectional area, the system may cause no trouble, but for ordinary working stress intensities, or for a strut of any importance whatever, no words can be too strong to condemn the principle, which is radically wrong, and no engineer worthy of the appellation will ever adopt it for a real strut. It will generally be found that

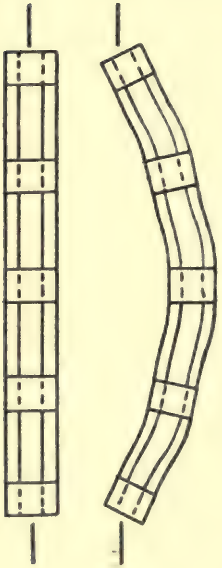


FIG. 67.

the weight of the batten plates is practically as great as, and sometimes greater than, that of a suitable triangulated system, and also that they require as many rivets. The batten plate system is supposed to be cheaper in construction. but that can only be, at the most, a very small percentage. Taking the reduction of strength into account it is very expensive.

An example of the inefficiency of this type of construction is afforded by the collapse, in December 1909, of a large gas-holder tank at Hamburg. This was traced to the failure of a steel strut supporting the floor of the tank. Some particulars of this strut are available. It was formed of two 6-inch by 3-inch channels, placed back to back, with a space of 1 inch between them. The whole strut was a little over 11 feet in length, and batten plates, 6 in. by 5·5 in. by 0·3125 in., connected the channels together at intervals of 40 in. There were therefore only two such points of connection, dividing the strut into three unbraced lengths. This strut was intended to carry a load of 118,000 lbs., and it is stated that some calculations which were made, using Euler's formula and treating the strut as if it had a solid web, indicated that there was a factor of safety of 4·5. The length of the equivalent free-ended (Type 2) strut was evidently taken as 0·7 of the total length. The strut collapsed under a load of 133,000 lbs., or exactly one-fourth of what its ultimate strength was supposed to be. Some of the batten plates were torn across the rivet holes, there being two rivets in each edge of each plate.

From a study of Fig. 67 it will be seen that what happened is exactly what might have been expected, and the whole circumstance confirms the foregoing remarks.

Diaphragms, etc.

If a long strut is built up, say, of four booms braced together (the section forming a square or rectangle, as in Fig. 68), the side bracings prevent the booms from bending individually in directions parallel to the axis X X or Y Y but, excepting the stiffness of the connections between the bracing bars and the booms, there is nothing to prevent the latter bending in diagonal directions, one pair moving inwards and the other outwards, as shown by the dotted lines.

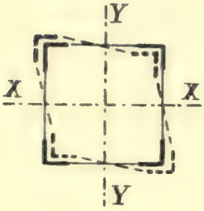


FIG. 68.

To prevent this distortion it is very usual to insert stiff transverse diaphragms at intervals in the length of the strut. The desired result is attained so far as relates to this form of distortion, but other conditions are introduced which are detrimental and often serious.

When fitted in a tubular or plate section column, the diaphragms are also supposed to prevent local buckling of the plates. Suppose that a diaphragm is fixed in a braced strut as at *a b* Fig. 69 (*b*). When the strut is loaded the diameter increases, but the expansion is prevented locally in the neighbourhood of the diaphragm. The booms then become distorted as shown by the dotted lines. In the same way a diaphragm inserted in a tubular column, as in (*a*), prevents expansion. Adjoining these diaphragms the strut is under conditions similar to those occurring at the ends in Fig. 61, and under heavy loads local buckling may be expected.

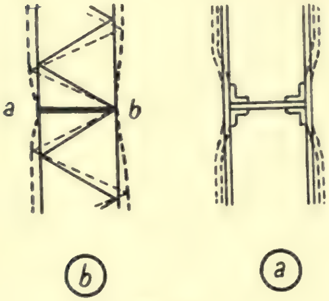


FIG. 69.

The amount of expansion in a tubular column or solid webbed strut is not so great as in a braced strut, because in the case of the solid section it is due to the actual lateral expansion of the material, while in the braced strut it is almost entirely due to the angular movement of the bracing bars when the panels shorten, and the spreading is greater. Owing to the comparative length of the panels in the braced strut, the rate of tapering will generally be less than in the tubular form.

That this objection is real and serious is strikingly illustrated by the results of column tests recently made in America, and described in "The Engineer" of March 24 and 31, 1911. In considering the re-building of the Quebec Bridge, after its collapse, the question of the construction of the compression members was carefully gone into. It was thought that one of the defects of the original design was a want of diaphragms in the main compression members. Certain new types of struts were devised as being better, and large scale models were made and tested. The section of these was of the form shown in Fig. 70 (*a*), the four main webs being made up of several plates laid together. To connect these webs bracing systems were fitted where shown by the dotted lines, and in the three positions shown by strong lines plates were fitted instead of bracings. These models were not toys, the largest of them measuring 20 ins. × 25 ins. × 36 feet in length and having a sectional area of 57 square inches, the model itself being a strut of considerable magnitude. Sixteen struts were tested, but four of them proved to be beyond the capacity of the testing machine. It is a very significant fact that eight of the twelve which were tested directly to destruction in the machine failed by local buckling of plates immediately adjoining the diaphragms, as in Fig. 70 (*b*), which represents a part longitudinal section at a transverse diaphragm; and that the remaining four failed by buckling of wings of angles, etc., near the same points. Therefore, the very diaphragms which were introduced to prevent local distortion were the cause of such distortion being set up and the prime cause of the first signs of failure.

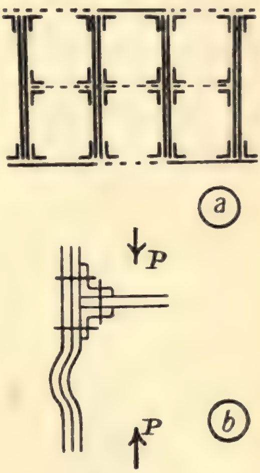


FIG. 70.

When a diaphragm is fitted into a tubular strut, or to one having a section with plate webs, if the outside was reinforced with extra plates, as indicated in Fig. 71, the bulging tendency would be considerably reduced, although not removed. The effect would still remain, to some extent, at the points *a a*, although the effect shown in Fig. 62 would be approximated to. This reinforcement is, however, not very practicable except in struts of comparatively large size.

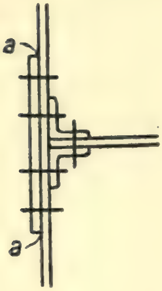


Fig. 71.

Diaphragms are quite unnecessary in braced struts, and the diamond-shape distortion can be prevented by connecting together, *at intervals*, the booms diagonally opposite by simple bars, for example similar to the bracing bars themselves. The same size as the bracing bars will be quite sufficient, as they will have less to do. If these diagonal bars were fitted transversely, at right angles to the axis of the strut, they would have the same objection as the diaphragm, but if they are fixed at an angle to the longitudinal axis, as in Fig. 72, the inclination being the same as that of the bracing bars on the sides, they will not prevent the strut from spreading, and they will be quite as effective, in preventing distortion, as if at right angles to the axis.

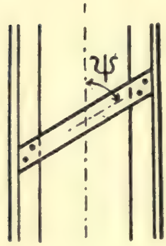


Fig. 72.

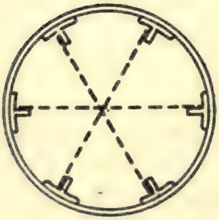


Fig. 73.

In a tubular or plated section it is not so easy or convenient to fit stretcher bars diagonally. In most of such cases if, instead of fitting diaphragms, a similar weight of metal were added to the main stress carrying parts, so as to increase the sectional areas throughout, the result would nearly always be more effective, as the stress intensities would be reduced. If such additional metal, in a tubular strut, were put into the form of longitudinal ribs, instead of diaphragms, as in Fig. 73, and these ribs connected together, at intervals, by bars fitted diagonally as in Fig. 72, the result would be still better. The angle of the diagonal bars would have to be made to suit the amount of expansion, or spreading, to be allowed for, so that each of them in its new position would remain unaltered in length. Such longitudinal stiffening ribs would also much assist in preventing the formation, at any part of the length of the strut, of wavy undulations of the nature shown in Fig. 70.

Torsional Stiffness.

It is necessary to ensure that, between points of support, a strut has a sufficient amount of torsional stiffness. A circular tubular strut has the greatest stiffness possible. Any other prismatic form, such as a square with members at the four corners and braced on all four sides has great stiffness. It is frequently stated that such a section as shown in Fig. 74 has the metal distributed to much advantage, but it will be seen that such a strut has very little torsional stiffness. The flanges of the projecting portions may be sufficient to resist very local distortion, but there is little to prevent the parts near the centre of an unsupported length from deflecting as indicated by the dotted lines, the centre sections retaining their form but rotating relatively to the end sections. Such deflection would relieve the extreme compression fibres, but throw the stress on to those nearer the central axis, and, on the whole, increase the stress intensities. Such struts frequently have diaphragms inserted to preserve their contour, but these have very little effect in preventing torsional deflection unless placed very close together, in which case a similar weight of material formed into bracings between the outer edges of the projecting parts would be very much more effective.

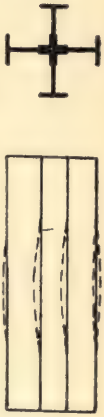


FIG. 74

Bending under own Weight.

If a strut, lying horizontally, is of any considerable size or has a high ratio of $\frac{L}{r}$ the bending moments due to its own weight may be great enough to cause a deflection sufficient to affect the resulting stress intensities to an appreciable extent. The amount of such deflection can be approximately calculated by ordinary methods as applied to girders, and should be taken as equivalent to an initial deviation of the axis from the line of the direct longitudinal force. The effects of deflection due to this cause may be partially counteracted by constructing the strut with an initial camber in the reverse direction to the deflection and to an equal amount, but in practice this is extremely difficult to carry out. Another method is to arrange the points of application of the end forces so as to lie below the axis, either by locating the centres of pin connections below, or arranging the intersections of the connecting members in the structure below the axis of the strut, or what comes to the same thing, and is probably the best method, the sectional area of the upper main member of the strut can be increased to carry the extra compressive stress due to bending under its own weight. This is, of course, equivalent to raising the axis of the strut above the line of the longitudinal forces.

A continuous compression member should be treated as a continuous girder in calculating deflection due to its own weight, although, as it has been shown, such continuity may not assist in resisting bending due to longitudinal forces.

If the strut is not horizontal, but inclined, the initial bending moments and deflection will be reduced precisely in the same way as in an inclined girder as compared with one lying horizontally.

If a horizontal or inclined strut should carry any other, dead or moving, loads the deflection due to the maximum combination of these must also be allowed for.

It sometimes happens that long braced struts in vertical working positions have to carry small loads, and only require very light bracings to resist the normal shearing stresses. In the course of manufacturing such struts, or in raising them into their working positions, they may at times be supported in such a way that temporary stresses come upon the bracing members, for example if lying horizontally and supported at only a few points, perhaps two, or when being lifted by slings passed round their middle parts and the ends overhanging. It must be ensured that such stresses are not high enough to cause any damage or permanent set in the strut. If necessary, the sizes of the bracing bars must be increased to meet the temporary stresses.

It must not be forgotten that in an inclined or horizontal strut shearing stresses, as well as bending moments, are produced by its own weight, and web rivets or diagonal bracing bars may have to be correspondingly increased. These shearing stresses are easily calculated by the ordinary rules applying to girders.

Wind Pressure.

In any exposed structure the struts, as well as the other members, may be subjected to side pressure from wind, and this will cause deflection, which must be allowed for. The only method of doing so is, in addition to calculating the stresses due to such bending, to treat the deflection as an initial deflection affecting the stresses due to the direct load. As the wind may blow in any direction, the deflection may also be in any direction if the strut is vertical. If horizontal, the pressure may come upon either side, or it may be sheltered in one direction.

As in the case of stresses due to the weight of a strut, the wind pressure will also cause shearing stresses affecting rivets and bracing bars.

From a practical point of view the amount of the wind pressure to be provided for has to be decided. The conditions of storms vary in different countries. In Britain there are Board of Trade regulations, by-laws of County Councils, and of other local authorities, which vary much, and which have to be complied with, whether reasonable or not. In some cases it is required that pressures up to 56 or 60 lbs. per square foot shall be provided for.

A question sometimes arises as to whether the maximum wind pressure is to be provided for in conjunction with the maximum ordinary load, and it is often difficult to decide. In the case of dead loads, which are permanently on the structure, the maximum combination must, of course, be provided for. In the case of moving loads, the condition may be totally different. Railway traffic, for example, could not be carried on in a storm violent enough to produce pressure of 56 or 60 lbs. per square foot, as many ordinary vehicles would be overturned before such a pressure was reached. In the case of a footbridge, traffic would cease under a less pressure, because at, say 30 to 40 lbs. per square foot, a man facing the wind would have to lean forward to an angle of nearly 45 degrees in order to maintain his balance, and obviously foot traffic would then be impossible.

In such circumstances the maximum wind pressure under which traffic, or working, could continue should be allowed for in conjunction with the ordinary loads, and the strength of the struts also investigated under the condition of maximum wind pressure combined with permanent loads only.

Under extreme emergency conditions of wind pressure it is generally permissible to allow for a higher maximum working stress intensity, but, here again, local and other conditions vary so much that no universal rules can be laid down. Generally speaking, if under working conditions a factor of safety of, say, 4 is provided for, it is quite safe and good practice to allow a reduced factor of 3, or even slightly less, under the emergency conditions. In any case, however, care must be taken not to allow stresses approaching too near to the elastic limit of the material employed.

In calculating the pressure on a strut due to the wind it is usual, in the case of a single braced web, or solid web, formed with flat surfaces, to take the full wind pressure as acting over the whole of the exposed surface ; in the case of a braced strut of double web, or box section, the full wind pressure on 1·5 times the front exposed surface, as the anterior surfaces are partially shielded, and in the case of round struts one half of the product of the wind pressure and the projected area of the exposed strut.

CHAPTER X.

SHEARING STRESSES AND BRACED STRUTS.

A VERY important part of the strut problem is the determination of the nature and amount of the stresses in the webs or bracings connecting the booms or members which carry the direct stresses. It is a question which is generally passed over in a very casual manner.

Compared with a girder of somewhat similar cross section, the stresses in the web of a strut are much less than in the girder, because in a girder the whole of the stresses in the booms pass into the web, while in a strut the web receives only part of the boom stresses, that is, the portions which are produced by bending action, and not the portions due to the direct action of the load. Although the web stresses are comparatively small it must not be thought that they are unimportant. On account of their smallness, any errors in computing the stresses become proportionately greater.

If a strut is formed of rolled ⊏ or H sections, or built up of rolled sections and plates in such a way that it has a web resembling a plate web in a girder, the question of the shearing stresses need not be gone into in detail. Any such web, even if made of a thickness no greater than just sufficient to resist local buckling, is certain to be more than sufficient to carry all the shearing stresses which can come upon it.

When the strut is formed of separate bars or booms, united by triangulated systems of bracing, it is necessary to have some rules for determining the minimum sizes of the members consistent with the general degree of safety required in the strut as a whole. It is surprising to note the differences in the quantity of material put, by different designers, into the bracings of struts which compare with each other in general dimensions and sections. Such differences sometimes amount to several hundreds per cent., and while some are undoubtedly too light, there is very often great waste of material. It is, of course, better to have a surplus of material than to run risks of failure, but at the same time, unnecessary weight means extra cost, if nothing else.

The only process which has been generally adopted for calculating approximately the stresses in the members of bracing systems is as follows:—

In Fig. 75, let (*a*) represent the elevation of a braced strut of Type 2, having external forces, P P acting at its ends, along the axis. Let (*b*) represent the curve of bending moments, due to the action of P the maximum moment being at the centre and $= Pv$ Let (*c*) represent the curve of bending moments on a beam of the same length L as the strut, and carrying a uniformly distributed load of an intensity w such as would be necessary to produce a bending moment, at the centre, of the same numerical value as Pv Such a load would produce reactions R R and consequently transverse shearing stresses of similar amounts, at the ends of the beam, and these would produce corresponding stresses in the bracing bars. Calculations are made on these lines, and the results taken as the actual stresses on the bracing bars.

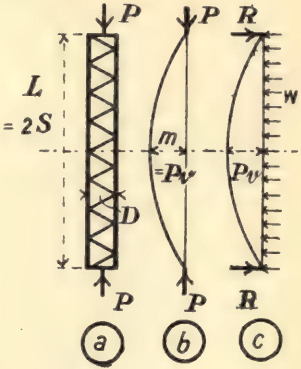


Fig. 75.

In making these calculations the value p'' is taken as from the equations from which the section was designed. Then, a being the sectional area of one of the booms, the total stress on each boom is $p''a$ and D being the distance between the centres of the booms connected by the bracings, the bending moment required to produce the stress $p''a$ is $p''aD = M$

In a beam with load uniformly distributed the bending moment is $M = \frac{w L^2}{8}$ therefore $w = \frac{8 M}{L^2}$ and substituting the value of $M = p'' a D$ this gives $w = \frac{8 p'' a D}{L^2}$

Then

$$R = \frac{w L}{2} = \frac{4 p'' a D}{L} \quad . \qquad (155)$$

or if the stresses on the section of the strut have been determined by using the Moment of Inertia, etc., we should have

$$M = \frac{2 p'' I_0}{\delta}$$

and then

$$w = \frac{8 p'' I_0}{\delta L} \quad \text{and} \quad R = \frac{4 p'' I_0}{\delta}$$

The total stress on the end diagonal is then taken as R × (cosec. of angle made by axis of diagonal with axis of strut).

If the system of bracing is double, or multiple, the total is divided uniformly among them.

Now, if the curve used was the true curve of moments produced by P and the intensity of w was adjusted, so that it not only gave the same maximum moment, but also a moment at any point in the length L the same as the actual moment produced by P the method just described would give practically exact results. If the true curve was used, it would be found that the intensity of the load w would not be uniform, but would vary from the centre towards the ends, and the value of the reaction R would not be the same as given by these equations. The assumption that w is uniformly distributed requires that the curve in (c) should be a parabola, and therefore the curve of bending moments is now assumed to be a parabola. Previously it had generally been assumed to be a sine curve. It is, in reality, neither one nor the other, and it should be clearly understood that this process is not exact, but merely an approximation, and simply resolves itself into a process of assuming a parabolic curve having a central ordinate of a certain height, and then finding the value of the tangent at the point where the curve meets the abscissa, any tangent to a bending moment curve giving the amount of the transverse shear at that point.

It may be, and it often is, argued that the actual amounts of deflection of struts, obtained in practice, are so small that it is of little importance, or of no importance at all, what curve is taken for the bending moments. It is a fact that the actual deflections are small, but it must not be forgotten that the values of the shearing stresses do not depend solely upon the numerical values of deflections, but chiefly upon the rates of increase or decrease in the direct stresses; in other words, they depend upon the form of the bending moment curve, and its relative rate of inclination throughout the length. The tangents are merely ratios and are quite irrespective of dimensions.

The real fact is that the determination of the form of the bending moment curve is the whole problem, and therefore instead of being of no importance, it is all important. No solution can be arrived at without finding, or assuming, a curve, and the properties of curves vary so much that it is necessary to at least know those of the true curve, so that the limits of error of any suggested approximation may be determined.

The assumption of a parabolic curve will, for example, always give a result over 27 per cent. greater than that obtained by the assumption of a sine curve. In both of these cases the result will vary directly as the amount of the deflection, while with the true curve the shear increases, at first, at a greater rate than the deflection.

In using the foregoing method the difficulty at once arises of finding the value of v which must be known before the bending moment can be ascertained. As Euler's formula, and all others of that type, represent that equilibrium is unstable immediately bending commences, it is impossible, by their use, to find any deflection at all corresponding to a given load. Modifications of these formulæ allowing for initial deflection do not show unstable equilibrium, but there is no ready method, by their use, of determining the correct amount of deflection for any particular case. The process of determining the deflection is often mere assumption, or guesswork.

In using the equations given herein the correct amount of deflection for any section can be found by equations (133) and (134). These give the maximum allowable deflection for any section corresponding to any given load and values of P and f The value of v thus obtained, when inserted in the equations for shearing stresses, enables the solution to be obtained.

It has been tacitly assumed that the curve of bending moments as distributed along the axis of a strut is the same as the curve of the bent strut itself. Take a straight line YXY_1 (Fig. 76) (*a*) of length L as representing the longitudinal axis of a straight strut, of Type 2, having forces, P P, acting along it, and applied at the ends. Let (*b*) represent the curve which the axis will assume when deflected under the load. This curve (*b*) is the curve of the bent axis but is not, in the ordinary sense, the curve of deflections, because the straight line YY_1 of the chord, or abscissa, is not of the same length as the axis, which is now the curve YXY_1 If the length of the curve L is divided into parts by points at a distance ds apart, and, as in (*c*), another curve is constructed with its abscissa of length L divided up in the same manner, and the ordinates at all points equal to those of the corresponding points in (*b*), then the curve (*c*) is the curve, according to ordinary defini-

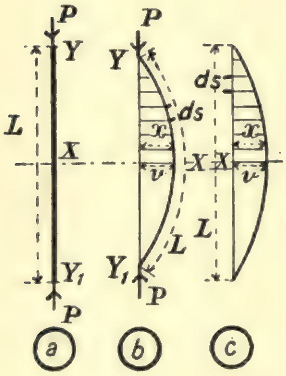


FIG. 76.

tions, of both deflections and bending moments, the abscissa $Y\,Y_1$ being the same as the originally straight axis. Curve (*b*) could only be taken as a curve of moments if the ordinates were considered in connection with distances measured along the curve instead of, as usual, along the abscissa. The form of curve (*c*) is the same as the result would be if the curve (*b*) were straightened without altering in length, while the abscissa became stretched out to form a new curve, all the ordinates remaining unaltered in length.

From this we see that, at any point in the deflection curve, the tangent of its inclination to the abscissa is the same as the sine of the inclination at the corresponding point in the curve of the bent axis. The ordinates of the bending moment curve are equal to those of the deflection curve multiplied by P, therefore the curve (*c*), to suitable scales, represents both deflections and bending moments. As the abscissa of the bending moment curve is the same as that of the curve of deflection, while the ordinates differ all in the same ratio, we have

$$\text{(ordinate of B.M. curve)} = \text{(ordinate of deflection curve)} \times \text{(constant)},$$

and therefore

$$\begin{aligned}\tan\text{ (B.M. curve)} &= \tan\text{ (deflection curve)} \times \text{const.}\\ &= \sin\text{ (curve of bent axis)} \times \text{const.}\end{aligned}$$

Now as the bending moment $= P\,x$ and x is the ordinate of both the curve of the bent axis and the curve of deflections, it follows that the constant $= P$ therefore at any point in the length of the axis of the strut, the transverse shearing force is

$$\begin{aligned}R &= P \times \tan\text{ (B.M. curve)}\\ &= P \times \sin\text{ (curve of bent axis)} = P \cdot \sin i \qquad (156)\end{aligned}$$

The equations to the moment curve corresponding to any curve of bending are nearly always of a very complicated nature, but when the curve of bending is known, it is not necessary, for the purpose of determining the shearing forces, to construct the moment curve, as, owing to this relationship of the tangent and sine, the value of P is easily determined directly from the curve of bending.

The Distribution of Shearing Stress under Ideal Conditions of Loading.

In Fig. 77, let (*a*) represent the true curve of the bent axis, for ideal loading conditions, and (*b*) the corresponding bending moment curve. The force P at Y may be resolved into a force T_Y acting tangentially to the axis at that point, and another force R_Y acting at right angles to the tangent. Then R_Y is the transverse shearing force at Y, and

$$R_Y = P \cdot \sin i_Y \qquad (157)$$

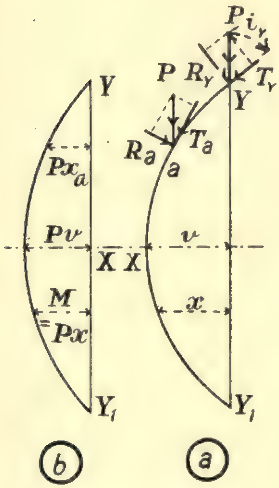


FIG. 77.

At any other point such as *a* the component (parallel to P) of the thrust along the axis is equal to P and therefore, resolving P into two forces, T_a and R_a tangential and normal to the axis, the transverse shearing force at that point is $R_a = P \cdot \sin i$. So that at any point in the curve, at distance *s* from X, we have

$$R = P \cdot \sin i \qquad (158)$$

This result is the same as equation (156).

In curve (*b*) distances along the abscissa from X are equal to *s* in curve (*a*). Any ordinate is the bending moment at that point, $M = Px$ In any bent bar the transverse shearing force at any point is equal to the tangent to the bending moment curve at that point. Therefore at *a* in curve (*b*), the tangent is

$$\frac{d\,M}{d\,s} = \frac{d\,(P\,x)}{d\,s} = P\frac{d\,x}{d\,s} = R$$

Now,

$$\frac{d\,x}{d\,s} = \sin i$$

therefore

$$R = P \cdot \sin i \quad \text{and} \quad R_Y = P \cdot \sin i_Y \qquad (159)$$

the same as by (157) and (158).

By (90)

$$\cos i_Y = 1 - \frac{v^2}{2\,K^2} \quad \therefore \sin i_Y = \left\{1 - \left(1 - \frac{v^2}{2\,K^2}\right)^2\right\}^{\frac{1}{2}}$$

and, substituting this in (157), we have the exact equation

$$R_Y = P\left\{1 - \left(1 - \frac{v^2}{2\,K^2}\right)^2\right\}^{\frac{1}{2}} \qquad (160)$$

This equation is slightly cumbersome, but we saw by (101) that $\sin i_Y = \sin(2\,\theta)$ and by (139) that the substitution of $2 \times \sin\theta$ for $\sin(2\,\theta)$ caused only a quite negligible error. Making the same substitution here, we have, approximately, from (157),

$$R_Y = 2\,P \cdot \sin\theta = 2\,P\frac{v}{2\,K} = P\frac{v}{K} \qquad (161)$$

independent of length

$K = \sqrt{\frac{EI_0}{P}}$

the error being about $\frac{1}{35{,}000}$, or 0·003 per cent., for comparatively large deflections.

$v = \frac{1}{8}($

$R_Y = \frac{P\sqrt{P}}{\sqrt{EI_0}} \cdot \frac{1}{8}\left(\frac{f}{p} - \frac{1}{A}\right)$

Distribution of Shearing Stresses under Practical Conditions of Loading, etc.

For practical conditions, where P is not truly an axial load, equation (137) gives the curves of bending, and it is only necessary to find the transverse components R for that curve.

At any point in the curve, the exact value of R is still given by equation (158). At the end of the strut this equation becomes

$$R_z = P \cdot \sin i_z \quad . \quad . \quad (162)$$

and the value of sin i can easily be found from equation (88), which becomes

$$\cos i_z = 1 - \frac{1}{2\,K^2}(v^2 - z^2)$$

P. 117 [illegible]

As a very simple and close approximation, equation (161) may be used, when we have, approximately,

$$R_z = P\frac{v}{K} \quad . \quad . \quad . \quad (163)$$

the justification for the substitution being the same as in arriving at equation (142). Owing to the small curvature of the deflection curve near its ends, it becomes sensibly coincident with the end tangent of the completed curve, as in Fig. 44, and the inclination at y_z differs little from that at Y The error caused by the substitution will be proportionately greater than it was in finding the length of the strut, but it is still a very small percentage, and negligible.

In considering the effect on the bending stresses, of initial departures from ideal conditions, the extreme case of a combination of the maximum deviation due to each disturbing cause occurring at the same time, was not allowed for, but a reasonably probable maximum combination ; and the same course should be adopted in providing for shearing stresses. It is very improbable that the maximum initial departure of the line of P from the axis will occur at the same time and in the same direction at all points in the length. Any such variation from a uniform deviation will cause irregularities in the deflection and bending moment curves, and these irregularities must cause variations of the inclination of the curve to the abscissa, the result being that the tangent to the moment curve will be greater at some points, and consequently the corresponding transverse shearing force R will also be greater than if the initial deviation was uniform.

In Fig. 78, let (*a*) be a strut subjected to external forces as shown. If the straight line X X parallel to P P were the axis, the curve of bending moments would be of the form *a b c* in (*b*). If, however, the axis X X did not follow the straight line, but the dotted line in (*a*), the curve of bending moments would take the form of the dotted line in (*b*), the point of maximum moment being then, not at the centre but at *d* towards one end. As an initial deviation has more effect near the centre of the strut than near the ends, and as any deviation due to mechanical causes is more likely to be greater near the centre, the probabilities are that the maximum deflection will take place within a certain distance from the centre, and it seems reasonable to expect that it will practically always

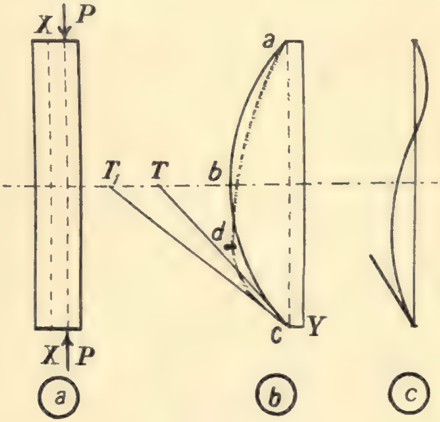


Fig. 78.

occur within the middle third of the length. Then, if $c\,T$ is the tangent at c to the curve $a\,b\,c$ the tangent $c\,T_1$ to $a\,d\,c$ will be at a greater inclination than $c\,T$ The increase in the inclination will be slightly greater than the ratio of $\frac{b\,c}{d\,c}$ and, if $b\,d$ is about one third of $b\,c$ it will be greater than $\frac{3}{2}$. (164)

This deviation towards one end is not hypothetical, but actually occurs in practice. The recorded deflections of struts under test show considerable variation in the distribution of deflection. It is not uncommon to find a deflection curve as indicated in (*c*), when, even though the maximum deflection or bending moment may be less, the tangent or slope of the curve, at some point, may be greater than for a more uniform curve. It is often found, during tests, that the bracings in struts fail in one direction near one end and in an opposite direction towards the other end.

Take, now, equation (163), for uniform deviation,

$$R_z = P\frac{v}{K} \quad \text{where} \quad K = \sqrt{\frac{E\,I_0}{P}}$$

We find, from (136) and (110), that

$$S = \sqrt{\frac{E\,I_0}{P}} \cdot \frac{\pi}{2} \text{ (very nearly)} = K\frac{\pi}{2}$$

$$\therefore \qquad K = \frac{2\,S}{\pi} \text{ (nearly)}$$

therefore, by substitution,

$$R_z = P \cdot \frac{v}{2} \cdot \frac{\pi}{S}$$

Then, to allow for longitudinal irregularities, by (164)

$$R_z = P \cdot \frac{v}{2} \cdot \frac{\pi}{S} \times \left(\frac{3}{2} +\right) = P \cdot \frac{v}{S} \times 2{\cdot}36 +$$

and, seeing that some allowance should be made for slight irregularities in fitting and securing the bracing bars, we may increase the constant from 2·36 to 2·5, when we get, allowing for the probable general maximum rate of departure from ideal conditions,

$$R_z = 2 \cdot 5 \frac{P v}{S} \quad . \quad . \quad . \quad (165)$$

Note that the value S is used in this equation and not s_z

It may be here noticed that the use of a parabolic curve with the same value of v would give an equation containing a constant 2, in place of 2·5, but it makes no allowance for irregularities in deflection. Again, the use of a sine curve, also without such allowances, would give a constant 1·57. The higher value of 2·5, adopted in equation (165), is not only justified, but proved necessary by the results of tests. It frequently happens that, under test, the bracing bars of struts fail before the main booms, or members, even although the sizes of bracings would satisfy the ordinary equations.

It should also be noted that the approximate equation (165) will not give correct results if used to determine shearing forces for varying loads. It is an approximation only to the ultimate stresses when the strut is about to fail. Shearing stresses do not increase in the same proportion as an increasing load. The rate of increase is small at first, but the rate increases as the load becomes greater.

Seeing that, under ideal or good practical conditions, the shearing forces are not uniform throughout the length of a strut, but become practically nil near the centre of Type 2, or about midway between points of contrary flexure in higher types, it might be thought economical to reduce the sizes of the bracing bars over some portions of the length. As we have seen, however, the points where the deflection is a maximum, and the shearing forces nil, are unknown and are certain to be different in each strut, and the sizes of the bars could therefore only be, at the most, slightly reduced in some places. In columns of moderate size the economy would be a doubtful point, as the bars at the points of greatest shear are already comparatively small, and from a practical point of view, a change of section might involve as much or more cost than any saving in material. It is usual to carry the same sizes of bracing from one end to the other, even in struts of large size, and it appears advisable to do so, unless there are some very unusual circumstances. In such an event each case must be considered on its own merits. No general formulæ can be given for even the probable limits of reduction in shearing stresses.

Braced Struts.

The curves of bending, etc., which we have considered are those for struts of solid construction, such as solid bars or rolled sections. If the strut is formed of booms connected by bracing, there is no axis, in the sense generally defined, where the longitudinal stresses are nil, and the diagram of bending moments will not be a continuous curve. Take a strut with bracing of the form indicated in Fig. 79 (*a*). When loaded,

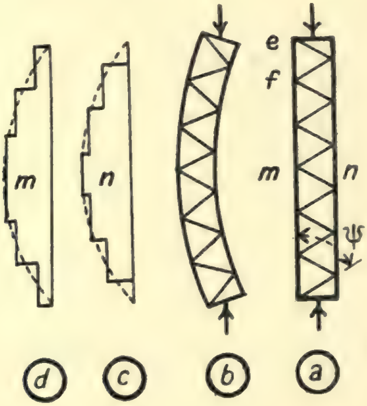


Fig. 79.

this will deflect to one side as indicated in (*b*). If, in (*c*) and (*d*), the dotted lines are the curves of moments for a similar strut with a continuous web, the moments for the braced strut will be of the stepped forms shown. These steppings will be constructed in a manner somewhat similar to the moment diagrams for a braced girder. Diagram (*d*) will represent stresses in members of the booms on the side *m* of the strut, and diagram (*c*) those on the side *n* The stress in one section of the boom (such as *e f*) cannot vary throughout its length as there are no web connections between *e* and *f* and in the same way the stress in each bay of each boom is constant over the bay. It will be found that the two diagrams (*c*) and (*d*) are not alike. The stresses in the two booms are not alike, and therefore the strains and curvatures are dissimilar. The differences between the stresses in the booms will vary with the type of bracing.

If a strut was constructed with only a small number of bays of bracing, such as four or six, it would be necessary to calculate the deflection and moments specially, if anything like accuracy was required. It is seldom, however, that there are less than ten, twelve, or even more bays of bracing, and then the differences become small enough to be neglected, and the ordinary curves may be used, and the bending stresses determined in the same way as for a webbed section.

To determine the stresses in the bracing bars, let ψ, Fig. 79 (*a*), be the angle between the axis of the end bar and the axis of the strut. Then the stress on

the bracing bar $= \dfrac{R_z}{\sin \psi}$. (166)

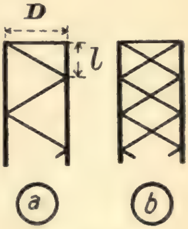


Fig. 80.

or, Fig. 80 (*a*), if l is the length on the boom subtended by the bracing bar, and D the distance between the booms, the length of the bar is $\sqrt{D^2 + l^2}$ and the stress on the bar

$$= R_z \frac{\sqrt{D^2 + l^2}}{D} \quad . \quad . \quad (167)$$

This is the same as equation (166), but is expressed in linear units.

If there are two or more braced webs, the value given must be divided equally among them, and if there should be a double system of bracing, as in Fig. 80 (*b*), the stresses will be halved, each system taking one half.

Systems of Bracing.

There are a number of different types of bracing which are commonly used in webs of struts. It is almost entirely a question of individual preference on the part of the designer which leads to the adoption of a particular type, rather than any question of efficiency. At the same time there are advantages in some of the types, and disadvantages in others. The usual simple types are illustrated in Fig. 81.

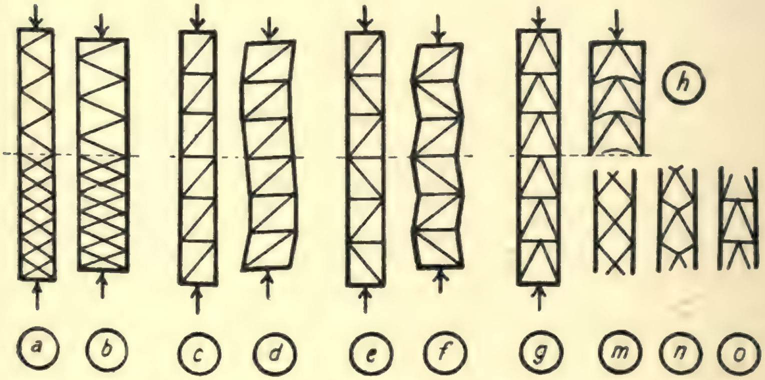


Fig. 81.

The type (*a*), similar to the well-known Warren bracing in girders, is the best of all. When the strut is loaded, it becomes shortened in length, and each of the bays is shortened, as indicated in (*b*). The bracing bars assume a slightly different angle, and the main booms are pushed out, and the whole strut expands in diameter. This takes place uniformly throughout the length and introduces no disturbances of the main stresses, except such as may be due to stiffness of bracing connections and which is common to all types.

If this system of bracing is doubled, as in the lower halves of the figures, the result is the same.

The type of bracing shown by (*c*) is that known in girders as the N type. When loaded, the bays shorten, as before, but owing to the non-symmetry of the bracing, the strut will not expand uniformly. The transverse members of the bracing tend to hold the booms together, and the diagonals tend to throw each of the originally rectangular bays or panels into a diamond shape, and the general result is to cause a distortion as indicated in (*d*). Of course these distortions are exaggerated in the diagrams, and it may be argued that they are very slight. That is quite true. They may be even microscopic, but so also are the distortions due to direct loading, and, as already explained, the distortion is the measure of the stress, and it cannot be neglected. In the case of struts the distortions are everything. This cannot be emphasised too much. Such irregularities of deflection cause a transfer of part of the direct stresses from one boom to the other in each half of the length of the strut, so that at some points the stresses are greater than if there was a continuous web. In the same way the direct action of P causes stresses in the diagonals, in addition to those due to bending. This is one reason for the necessity of the increased constant in equation, and one argument against making any reduction in the sizes of the bracing bars at any points.

Fig. 81 (*e*) shows a type of bracing which is the same as (*c*), with every second diagonal reversed. It will be easily seen that the effect of shortening under load will cause distortion alternately to the right and left in each succeeding bay, as indicated in (*f*) The distortions in successive bays are not cumulative as in (*d*).

Another type of bracing which has been recently introduced and sometimes called the K type, is shown in Fig. 81 (*g*). It is difficult to see what advantage there is in this type, unless the provision of a straight transverse member serves to simplify any side connections, but such connections are unusual in a strut. This type is somewhat the same as the double system, shown in (*a*), the intersection of the crossed bars being raised as in (*m*), (*n*) and (*o*), until the two upper halves of the two bars are in a straight line. The stresses in the corresponding portions of the bars are of the same sense, but the K type is not so economical, requiring a greater amount of material. From an economical point of view it has nothing to recommend it and it is anything but pleasing in appearance. The effect of a shortening of the strut under load is shown in (*h*). As the bays shorten, the diagonal struts meeting at the centre of the straight tranverse member bend the latter, causing secondary stresses and considerable local disturbances.

There is still another type of bracing, and one which is perhaps used more than any other, and it is a type which calls for considerable special consideration. It is illustrated in Fig. 82, and consists of a series of straight transverse bars

no
think

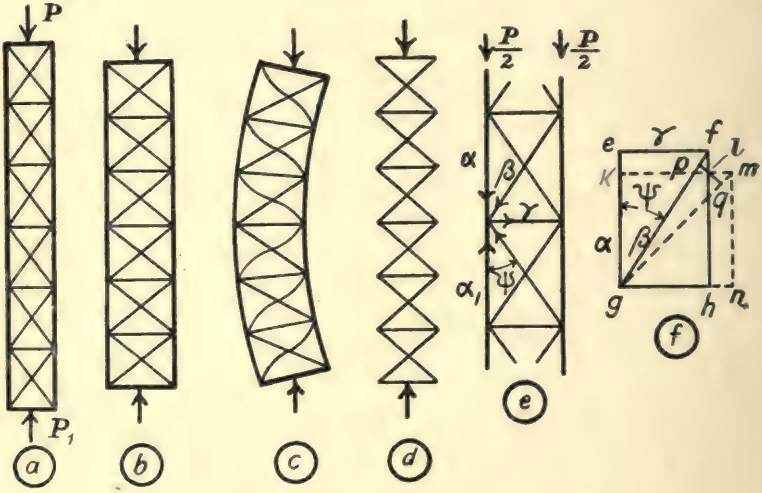


FIG. 82.

connecting the booms and forming a number of rectangular panels, in each of which two crossed diagonal members are placed.

When the column shortens under load it becomes as in (*b*). If the diagonal members are stiff they tend to spread the booms laterally, but the transverse members resist the spreading. There are then compressive stresses in the diagonals and tensile stresses in the transverse members. The strut expands laterally but to a less extent than if the transverse members were omitted. At the end of the strut the diagonals carry stresses having longitudinal components opposed in direction to the external forces P P_1 and therefore assist the booms in carrying the load. The bracing bars are stressed by the direct action of P as well as by the bending action due to deflection of the whole strut.

as in
[illegible]
[illegible]

If the diagonals are merely tie rods, flat bars, or of any other form incapable of resisting compressive stress, they become slack under load, unless set up with a sufficient initial tension, which in the case of riveted or pin connections is impossible, while if they are adjustable screwed rods it is impossible to set them all up uniformly. If they become slack there is no resistance to prevent the whole strut bending sideways, which it will do, as in (*c*), until one set of the diagonals becomes taut, while the other set will buckle and remain unstressed. The effect is then the same as if the strut had an initial deflection or curvature, and the action of P tends to increase the deflection, and so on. The extent of this bending will generally be considerable, and the fact of its existence indicates a defect in the system.

If the diagonals can resist compressive stresses, the whole of the bracings, apart from the booms, are as shown in (*d*), and form in themselves a system capable of carrying a load applied in the direction of P without collapsing. When combined in the strut, the bracing system, and the booms, are then two separate load-carrying systems acting jointly, and each will carry a share of P in proportion to the resistance which it exerts against being shortened in length.

In Fig. 81, (*e*), is shown a portion of such a system. The

82

load P if resisted equally by each half of the system, will cause a thrust $\frac{P}{2}$ along the axis of each of the booms. Let α, β and γ be the sectional areas of the members carrying these letters. Let ψ be the angle between a diagonal and the longitudinal axis.

When the strut becomes shortened, each bay or panel will shorten and at the same time tend to expand owing to the transverse components of the compressive stresses in the diagonals. This spreading tendency will be resisted by the transverse members, which will be subjected to tensile stresses. The spreading will not be prevented, but will continue until the tension in the transverse members equals the transverse component of the stresses in the diagonals.

Then, in any single panel, if, as in (f), $efgh$ is the original shape, it will alter, under the action of P to $kmgn$ the diagonal gf then taking the position gm The angle ψ represented by egf will alter to egm but the change will always be so slight that both of these may be considered equal to ψ Any differences in the values of the sine or cosine are quite negligible, and the short arc plq may be taken as straight and at right angles to the diagonal.

Now, let the change in length of α ($=$ ek or fl) be $\Delta\alpha$ and similarly, the change in length of γ ($=$ hn or lm) be $\Delta\gamma$. If the panel shortened without spreading, fp would be the change in length of β call this $\Delta_1\beta$. If the panel then spread, qm would represent the corresponding change in the length of β call this $\Delta_2\beta$ The net result of a combined shortening and spreading will be

$$\Delta\beta = \Delta_1\beta + \Delta_2\beta$$

We then have, first, due to the shortening alone

$$\text{length } \beta = \frac{\text{length } \alpha}{\cos\psi} \quad \text{and} \quad \Delta_1\beta = \Delta\alpha \cdot \cos\psi$$

$$\text{strain in } \alpha = \frac{\Delta\alpha}{\text{length } \alpha} \quad \therefore \quad \Delta\alpha = \text{strain } \alpha \times \text{length } \alpha$$

strain in β

$$= \frac{\Delta_1 \beta}{\text{length } \beta} = \frac{\Delta \alpha \cdot \cos^2 \psi}{\text{length } \alpha} = \text{strain } \alpha \cdot \cos^2 \psi \quad (168)$$

We have, next, due to the spreading alone

$$\text{length } \beta = \frac{\text{length } \gamma}{\sin \psi} \quad \text{and} \quad \Delta_2 \beta = \Delta \gamma \cdot \sin \psi$$

$$\text{strain } \gamma = \frac{\Delta \gamma}{\text{length } \gamma} \quad \therefore \quad \Delta \gamma = \text{strain } \gamma \times \text{length } \gamma$$

strain β

$$= \frac{\Delta_2 \beta}{\text{length } \beta} = \frac{\Delta \gamma \cdot \sin^2 \psi}{\text{length } \gamma} = \text{strain } \gamma \cdot \sin^2 \psi \quad (169)$$

Combining (168) and (169), and noting that the strain in the diagonal due to the spreading is of the opposite sense to that caused by the shortening, we have the net result

$$\text{strain } \beta = \text{strain } \alpha \cdot \cos^2 \psi - \text{strain } \gamma \cdot \sin^2 \psi \, . \quad (170)$$

Now, as α and α_1 are in the same straight line, the stress in γ must be equal to the sum of the transverse components of the stresses in β and β_1 We then have the following relations :—

$$\text{stress } \alpha = \text{strain } \alpha \cdot \text{E} \cdot \alpha$$
$$\text{stress } \beta = \text{strain } \beta \cdot \text{E} \cdot \beta$$
$$\text{stress } \gamma = 2 \cdot \text{stress } \beta \cdot \sin \psi$$

$E = \frac{\text{stress}}{\text{strain}} \therefore \frac{\text{stress } \alpha}{\alpha \cdot \text{st}}$

? assumes stress $\beta = \beta_1$

$$\text{strain } \gamma = \frac{\text{stress } \gamma}{\gamma \cdot \text{E}} = \frac{2 \cdot \text{stress } \beta \cdot \sin \psi}{\gamma \cdot \text{E}}$$

$$= \frac{2 \cdot \text{strain } \beta \cdot \beta \cdot \sin \psi}{\gamma}$$

Substituting this in (170), we get

$$\text{strain } \beta = \text{strain } \alpha \cdot \cos^2 \psi - \frac{2 \cdot \text{strain } \beta \cdot \beta \cdot \sin^3 \psi}{\gamma}$$

and therefore

$$\text{strain } \alpha \;=\; \frac{\text{strain } \beta \left(1 + \dfrac{2 \cdot \beta \cdot \sin^3 \psi}{\gamma}\right)}{\cos^2 \psi} \quad . \quad (171)$$

Now

$$\tfrac{1}{2} \text{P} \;=\; \text{stress } \alpha + \text{stress } \beta \cdot \cos \psi$$

if β is end
panel

$$= \;\text{strain } \alpha \cdot \alpha \cdot \text{E} + \text{strain } \beta \cdot \beta \cdot \text{E} \cdot \cos \psi$$

$$= \;\frac{\text{strain } \beta \left(1 + \dfrac{2 \cdot \beta \cdot \sin^3 \psi}{\gamma}\right)}{\cos^2 \psi} \cdot \alpha \cdot \text{E} + \text{strain } \beta \cdot \beta \cdot \text{E} \cdot \cos \psi$$

$$= \;\text{strain } \beta \cdot \text{E} \cdot \left(\alpha \, \frac{1 + \dfrac{2 \cdot \beta \cdot \sin^3 \psi}{\gamma}}{\cos^2 \psi} + \beta \cdot \cos \psi \right)$$

Therefore,

$$\text{strain } \beta \;=\; \frac{\tfrac{1}{2} \text{P}}{\text{E} \left(\alpha \, \dfrac{1 + \dfrac{2 \cdot \beta \cdot \sin^3 \psi}{\gamma}}{\cos^2 \psi} + \beta \cdot \cos \psi \right)}$$

and

$$\text{stress } \beta \;=\; \frac{\tfrac{1}{2} \text{P}}{\dfrac{\alpha}{\cos^2 \psi} \left(\dfrac{1}{\beta} + \dfrac{2 \cdot \sin^3 \psi}{\gamma} \right) + \cos \psi} \quad . \quad (172)$$

Then

$$\text{stress } \gamma \;=\; 2 \cdot \text{stress } \beta \cdot \sin \psi \quad . \quad . \quad . \quad (173)$$

and

$$\text{stress } \alpha \;=\; \tfrac{1}{2} \text{P} \;-\; \text{stress } \beta \cdot \cos \psi \quad . \quad . \quad (174)$$

It will be seen that these stresses cannot be calculated from statical considerations. They depend upon the elasticity and deformation of parts under stress. If the sectional area of either the diagonal or transverse members is made greater, the total stresses carried by them will also be increased. This may, at first, appear paradoxical but a consideration of the diagrams will show that it is so. An examination of equations (172) and (173) will show the same fact, because if in equation (172) the value of either β or γ is increased, the resulting value of the stress is increased. An increase in the sectional area of all the bars will, when they are small compared with the booms, cause an almost similar rate of increase in the total stress on each bar, and consequently the stress intensity will remain practically the same. If the transverse members, alone, are increased in size, the stresses on the diagonals will be increased, while if the transverse bars are reduced the diagonals will be relieved, and if the transverse bars are removed the stresses on the diagonals (due to the direct action of P) become nil, the bracing bars then becoming of the type of Fig. 82, (*a*), lower half. In the majority of cases, where the diagonals can carry compressive stresses, the struts would be much improved if the transverse members were entirely removed.

It will generally be found that if ψ is about 30° and if the diagonal and transverse bars are of the same section, the stress intensity in each of them is about $\frac{2}{3}$ of the intensity in the boom. If ψ is about 45°, and the bars are alike, the intensity in the diagonals will be about $\frac{1}{3}$, and in the transverse bars about $\frac{1}{2}$, of that in the booms. When ψ is greater than 30° the total stresses in the transverse bars will always be greater than in the diagonals.

It will probably be surprising to most of those who look into this matter to find how great these stresses are. They are often greater than the stresses due to normal bending. There are great numbers of struts continually being constructed of this type without a thought for such stresses, although they are too great to be neglected without considerable risk. Their importance will be better realised on a reference to Chapter XII., where it is shown that the chief factor in the cause of collapse of the Quebec Bridge was the omission to provide for them in the bracings of the main compression booms of the cantilevers.

CHAPTER XI.

STRUTS WITH VARYING SECTIONS OR IRREGULAR LOADING.

Section not Uniform throughout.

THE whole of the preceding considerations apply only to struts in which the section is uniform throughout, and therefore all lines parallel from end to end. As, in that case, the maximum stress intensities diminish towards the ends, it might be thought advisable to reduce the sectional area correspondingly. If this was done it would be found that the curve of bending would not remain the same, but the curvature would be increased towards the ends, and consequently, under the same load, the deflection also increased. This would necessitate either the load or the length being reduced to avoid increase in the maximum stress intensity.

There are many ways in which such a tapering off towards the ends of a strut may be effected. In a braced strut the diameter or distance between the booms might remain constant while the sectional areas of the booms were reduced, which, in a built up strut would have to be done in steps; or the sectional area of the booms might remain constant while they were brought closer together, or the diameter of the strut reduced, towards the ends. In a solid column the diameter might be reduced towards the ends, either normal to one axis-plane only, or in all directions. In a hollow or tubular strut the diameter might remain constant while the thickness of metal was reduced, or the thickness of metal might remain constant while the diameter was reduced towards the ends, or there might be a combination of both methods.

Several of the older mathematicians spent considerable time in trying to solve the problem of finding the most economical form for the shaft of a column. The problem is of such intricacy as to be almost impossible of exact solution. The conclusion generally come to was that it is doubtful if anything can be gained by departing from the cylindrical form with a parallel shaft.

The methods of tapering and conditions of loading vary so much that it is impossible to do more than consider a few cases. The general effect will be realised on a consideration of Fig. 83. Let *a b c d* be the axis of a straight strut under ideal conditions of loading and of Type 2. Let the section be uniform and the same between *a* and *b* as between *c* and *d* while the section between *b* and *c* is greater, but uniform between these points. Under a load P the axis will bend as indicated by the curve in (*b*). The portion *a b* will be the same as a part of the curve of a strut of Type 2 of same section throughout, of the length *a b f* and bent under the load P In the same way, *c d* will be a portion of a curve *g c d* The centre portion *b c* will be the same as a portion of the curve of a strut of Type 2, of uniform section throughout, of length e *a b* X *c h* and bent under the load P The curves *e b* X *c h* and *a b f* touch each other at *b* and the curves *e b* X *c h* and *g c d* touch at *c* having common tangents at these points. The deflection at the centre is represented by v

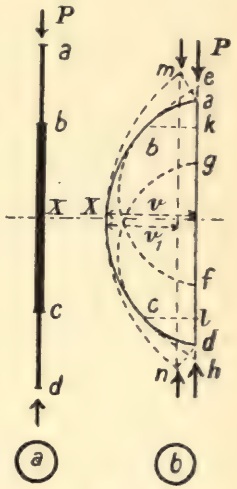


FIG. 83.

Now, if the strut was of the same section throughout (as between *b* and *c*) under the same load the curve of the axis would be as represented by the curve *m* X *n* of the same length as *a b* X *c h* and the deflection at the centre would be as v It will be readily seen that v must be greater than v_1 for a given load P From the equations given in Chapter VI., such a problem, although tedious, can be solved without very great difficulty. Knowing the section between a and b the curve $a\,b\,f$ can be found and the ordinate $b\,k$ (to the point b) calculated ; then, knowing the section between b and X that portion of the strut is in the condition of one of Type I with the load applied at a distance $b\,k$ from the axis, and v can therefore be determined.

A common form of tapered strut is that shown in Fig. 84. It is formed of curved booms connected by bracing bars, the section of the booms being uniform throughout and the distance between them reduced towards the ends. In such a case, everything depends upon the curvature of the booms. Let $a\,b$ (Fig. 84) be such a strut. Under the load P the curve of the bent axis will be as a X b in (b), while, if the strut was parallel and the section the same throughout, the curve would be as m X n For the tapered strut the deflection is v, against v_1 for the parallel form. In this case a solution cannot be arrived at as in the previous case because the change in the section takes place continuously instead of intermittently.

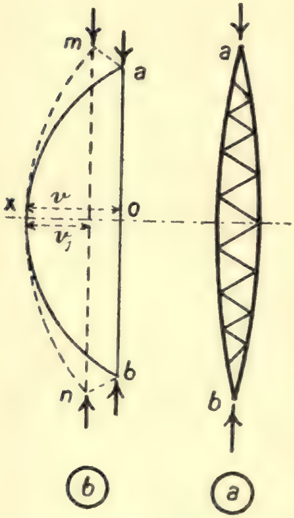


Fig. 84.

For any given form of tapering, the solution of the curve of bending, etc., is almost impossible, but any curve can be taken for the curve of bending at the instant of failure of the strut, or under any given conditions, and the form of the strut corresponding to that curve can be determined, and a general idea obtained as to the results of tapering. Compared with the parallel strut, the curve of bending of a tapered strut of this form must have relatively greater curvature towards the ends on account of the decreasing moment of inertia of the section. The curvature must, in fact, increase towards the ends instead of decreasing as with the parallel form. In the following example such a curve has been selected, and the form of the contour of the corresponding strut calculated. It has been found that a portion of an ellipse gives the desired results.

Example.

The required strut is to have a curved tapered form as in Fig. 85, and to have a section rectangular in form and made up of four bars, one at each corner, connected by systems of bracings.

The following data are given :—

Sectional area of each bar = 3 sq. in.
Total length of strut, L = 964 in.
Conditions of loading, etc. = Type 2.
Ultimate strength of material, f = 50,000 lbs. per sq. in.
Ultimate load, or strength, P = 180,000 lbs.
Modulus of Elasticity, E = 31,000,000 lbs.

Then

$$\text{A} = \text{total area} = 3 \text{ in.}^2 \times 4 = 12 \text{ in.}^2$$

$$p' = \frac{\text{P}}{\text{A}} = \frac{180{,}000}{12} = 15{,}000 \text{ lbs. per sq. in.}$$

$$p'' = f - p' = 50{,}000 - 15{,}000 = 35{,}000 \text{ lbs. per sq. in.}$$

Let it be taken that deflection will take place about the axis X X As the bars are comparatively small, take δ as the distance from the axis to the centre of gravity of the bar. Then, approximately,

$$\text{I}_0 = 4 \times (3 \text{ in.})^2 \times \delta^2$$

In Fig. 85, the curve $Y d X g Y_1$ is a semi-ellipse, a being the semi-axis major, b the semi-axis minor, and e the eccentricity. The portion $d X g$ of this curve is taken as the curve of bending at the instant that the maximum stress intensity f in the main members of the strut reaches the ultimate limit of 50,000 lbs. per sq. in. The line $d g$ is the base line from which ordinates to the deflection curve $d X g$ are measured, and this line is taken parallel to the major axis of the ellipse and at a distance from it $= h$

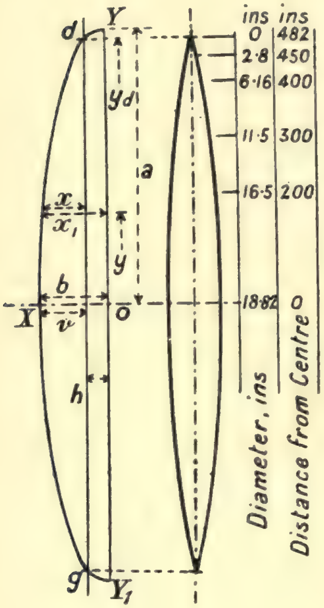


FIG 85.

For the ellipse, a is taken as $=$ 500 in.; $b = 30$ in.; $h = 8\cdot 1$ in. It will then be found that y_d the distance from O along O Y to a perpendicular through d or the distance of d from the axis O X is 482 inches the half length of the strut. For the present purpose this is measured along the chord instead of along the curve and is quite near enough for the purpose.

Then

$$e = \frac{\sqrt{a^2 - b^2}}{a}$$

and at any point in the elliptical curve the radius of curvature is

$$\rho = \frac{(a^2 - e^2 y^2)^{\frac{3}{2}}}{a\,b}$$

We have now to find the diameter $D = 2\delta$ of the strut at various points in its length, which will give it the proportions necessary to make the axis bend into the curve $d X g$ under the load P.

Consider first the centre section. When $y = 0$

$$\rho \text{ becomes } \frac{a^2}{b} \quad \text{and} \quad = \quad 8333 \text{ in.}$$

By (77)

$$\frac{1}{\rho} = \frac{p''}{E\,\delta} \qquad \therefore \quad \delta = \frac{p''\,\rho}{E}$$

and for the central section this gives

$$\delta = \frac{35{,}000 \times 8333}{31{,}000{,}000} = 9{\cdot}41 \text{ in.}$$

wherefore

$$D = 2\,\delta = 18{\cdot}82 \text{ in.}$$

Then

$$I_0 = 4 \times 3 \times \delta^2 = 1060 \text{ in.}$$

$$M = \frac{E\,I_0}{\rho} = \frac{31{,}000{,}000 \times 1060}{8{,}333} = 3{,}942{,}000 \text{ in.-lbs.}$$

and, as $M = P\,v$ we have

$$v = \frac{M}{P} = \frac{3{,}942{,}000}{180{,}000} = 21{\cdot}9 \text{ in.}$$

which, added to h gives $21{\cdot}9 + 8{\cdot}1 = 30$ in., and coincides with the value of b selected for the elliptic curve.

Now consider a section 200 in. from the centre.

We have

$$y = 200, \quad \text{and } \rho = \frac{(500^2 - 0{\cdot}996^2 \times 200^2)^{\frac{3}{2}}}{500 \times 30} = 6480 \text{ in.}$$

When $y = 200$, from the equations to the ellipse, we have

$$x_1^{\,2} = \frac{b^2}{a^2}\,(a^2 - y^2) = \frac{30^2}{500^2}\,(500^2 - 200^2) = 756,$$

therefore

$$x_1 = \sqrt{756} = 27 \cdot 5 \text{ in.}$$

and

$$x = x_1 - h = 27 \cdot 5 - 8 \cdot 1 = 19 \cdot 4 \text{ in.}$$

Then

$$M = Px = 180{,}000 \times 19 \cdot 4 = 3{,}490{,}000 \text{ in.-lbs.}$$

$$I_0 = \frac{M\rho}{E} = \frac{3{,}490{,}000 \times 6480}{31{,}000{,}000} = 730 \text{ in.}^4$$

and the value of δ required to give this value of I_0 is $\delta = \sqrt{\frac{I_0}{A}}$ taking δ as being (as it is in this case) sensibly equal to the radius of gyration; therefore

$$\delta = \sqrt{\frac{730}{12}} = 8 \cdot 25 \text{ in.}$$

and

$$D = 2\,\delta = 16 \cdot 5 \text{ in.}$$

Proceeding in the same way, we may obtain values of D for as many other sections as necessary. These have been calculated for $y = 0, 200, 300, 400$ and 450 in., and the dimensions are given in Fig. 85.

The curve of bending (*b*), and the derived elevation (*a*), of the strut are drawn to scale in Fig. 85, but, on account of the comparative smallness of the diameters, etc., in order to make the curves distinct, all the ordinates which are horizontal in the figure, are four times as great as their true dimensions, in proportion to the length, that is, the horizontal scale for (*a*) and (*b*) is four times as great as the vertical scale. If drawn to a natural scale the curves would not be quite of the same form as when the scale is distorted, but the difference is inconsiderable.

Comparing this strut with a parallel strut, having the same section throughout, as the central section of the tapered strut, we have, by (127) and (135)

$$L = 2S = \sqrt{\frac{E\,I_0}{P}} \times \pi$$

$$= \sqrt{\frac{31{,}000{,}000 \times 1060}{180{,}000}} \times \pi = 1344 \text{ in.}$$

that is, for the same load P, the length could be increased to that dimension. Or, if we take a parallel strut of the same central section, and the same length as the tapered strut, we have, by (135)

$$P = \frac{E\,I_0}{L^2} \times \pi^2$$

$$= \frac{31{,}000{,}000 \times 1060}{964^2} \times \pi^2 = 350{,}000 \text{ lbs.}$$

that is, for the same length the load could be increased to that amount.

Thus, we find that, *for equal loads* the parallel strut can be made

$$\left.\begin{array}{l} \dfrac{1344}{964} = \text{say, } 1 \cdot 4 \text{ times the length of the tapered strut.} \\[2ex] \text{Or the tapered strut must not be more than} \\[2ex] \dfrac{964}{1344} = \text{say, } 0 \cdot 72 \text{ times the length of the parallel strut.} \end{array}\right\} \quad (175)$$

In the same way, *for equal lengths*, the load on a parallel strut may be

$$\left.\begin{array}{l} \dfrac{350{,}000}{180{,}000} = 1 \cdot 94, \text{ or about twice the load on the tapered strut.} \\[2ex] \text{Or the load on the tapered strut must not be more than} \\[2ex] \dfrac{180{,}000}{350{,}000} = 0 \cdot 52, \text{ or about half the load on the parallel strut.} \end{array}\right\} \quad (176)$$

It will be seen that the curves of the sides of the strut in Fig. 85 are practically circular arcs, and its form is a very usual one for a crane jib or a derrick strut. Many thousands of such struts have been constructed in accordance with the common formulæ for parallel struts, and their Factors of Safety are therefore little more than one-half of what they are supposed to be.

The above investigation must not be misunderstood. The example is only worked out for a particular set of conditions and may be taken as a fair approximation to the ratios of length and strength for that type of strut. An alteration in the contour of the elevation of the strut may make a considerable difference. For example, struts are sometimes made parallel near the centre for about half the length and then formed with straight tapers towards the ends, and such a strut would be somewhat stronger than the curved one in the example, and the shorter the length of the end tapering, the greater the strength. The example gives a standard for comparison.

In this example the conditions have been taken as ideal. The same practical difficulties would however arise, and the same ratios of load and length may be taken as applying also to practical conditions.

Sometimes struts of this nature are tapered in all planes and sometimes only in one. If tapered relatively to both axes X X and Y Y so that the section remains of the same form throughout, although varying in dimensions, the same conclusions as to the worst direction of bending, etc., may be accepted as for the parallel strut; thus, if the section was square the worst direction would be along a diagonal, and so on.

If tapered in one plane only, or more in one plane than in others, the conditions become more similar to those of a parallel strut, with a section having I_0 greater in one direction than in another, but the conditions vary so greatly, and the processes of investigation are so intricate that no definite rules can be stated.

In a tapered strut the transverse shearing forces due to bending are, to some extent, counteracted by the transverse components of the stresses in the main booms, which are inclined to the axis. This is exactly the same as in the case of a girder, in which the depth is reduced towards the ends. In a girder tapered in a parabolic curve and carrying a uniform load, there are no stresses in the web bracings, and similarly, in a strut, the contour might be found so that the transverse stresses due to bending were all exactly taken up by the booms. There would still be a necessity for web bracings, because the direct load P tends to spread the booms of the strut, their axes being inclined to the main axis. Such bracings would, if of plain systems, all be in tension, but having regard to local irregularities, matters of transit, handling during erection, and abnormal conditions, which sometimes occur, anyone designing tapered struts will be well advised not to reduce the web bracings below what would be required for a parallel strut of the same central section and same length.

It may be pointed out that in the example although, for convenience, an elliptic arc has been taken for the curve of bending under the ultimate load, the curve would not remain elliptic for less loads, or for greater loads if such were possible. The elliptic form would only be a singular value of the general equation to the curve of bending, and it would only occur under these particular conditions. Any curve whatever may be taken as a curve of bending and the corresponding contour of a strut found to correspond, but in each case it would be only a singular value, unless it so happened that its equation was itself the general equation.

In the case of any tapered strut of great importance it would be possible, although extremely tedious, to find, by a tentative process, an approximate curve of bending corresponding to any given form of taper, but even then only for one particular set of conditions.

Section larger at one end of the Strut than at the other end.

Struts are sometimes to be seen tapering continuously from one end to the other. In such a case the curve of bending is not the same as in a parallel strut. The curvature is greater near one end than the other. It is seldom that such struts are formed with a large rate of taper, and although the equations for determining the section are not strictly applicable they may still be used without sensible error, as the deflection at the centre will not differ much. The extra stiffness at the larger end will generally quite compensate for the reduction at the smaller end. The section of greatest bending moment will not be at the centre of the length but somewhat nearer to the smaller end.

If such a strut has a comparatively large ratio of $\frac{L}{r}$, and is of the braced type, it may be found that the longitudinal shearing forces near the larger end are greater than near the smaller end because, although the rate of increase, or tangent to the bending moment curve, is greater at the smaller end, the negative inclination of the boom at the larger end may more than balance it, and the rate of increase of stress be greater at the larger end.

The Compression Booms of Girders.

The compression boom of every girder has to act, to some extent, as a strut. If the girder is of moderate size, the boom may be without lateral support from end to end and depend entirely upon its own stiffness to keep it from buckling sideways. Here, again, no universal rule can be given for determining whether the stiffness is sufficient or not. Each case must be treated separately.

Let Fig. 86 represent a simple braced girder of the Warren type, of uniform depth throughout, the compression boom, *a d*, being divided into three bays. Let the girder be loaded with equal weights *m* and *n* at *b* and *c*. The compression boom receives increments of stress P_a and P_b at *a* and *b* and similar increments P_c and P_d at *c* and *d* respectively, as indicated in (*b*). If this boom has no lateral support, it will

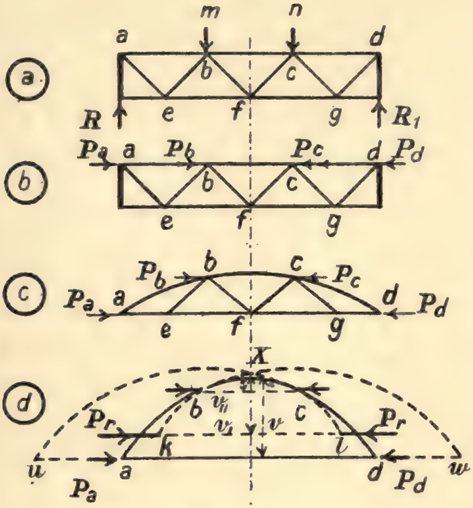


FIG. 86.

tend to bend sideways, and if it bends to one side, a plan view, looking down on the top of the girder, will be as represented in (*c*), the end posts being supposed to remain upright. The straight line *a d* represents the lower, or tension, boom which, owing to the nature of its stresses, is in stable equilibrium and has no tendency to become curved. The curve *a b c d* represents the form assumed by the upper or compressive boom. The diagonal members assume the positions indicated by their respective letters.

The increments of stress, P_a P_b etc., continue to act in directions parallel to *a d* The curve of bending is shown to a more distorted scale in (*d*). The result is the same as in a strut of length *a b c d* carrying loads P_a and P_b and their corresponding reactions P_c and P_d, at the points indicated.

At the centre point X of the strut, the bending moment is $P_a \cdot v + P_b \cdot v_{11}$ and this equals $P_r \cdot v_1$ if P_r is the resultant of P_a and P_b The centre portion *b c* of the boom is then in the condition of a portion of a strut of Type 2, of length *k b* X *c l* and bent to that curve under loads P_r applied at its ends, along the axis. The portion *a b* is in the condition of part of a strut of Type 2, of length *a b w* carrying a load P_a and reaction P_d at its ends. The portion *c d* is under similar conditions. In such a simple case the curves, moments, etc., can be calculated, but if, as is usual, the girder is divided into a number of bays much greater than three, the problem becomes more intricate.

If we take a plain webbed girder, as in Fig. 87 (*a*), carrying a uniformly distributed load, and having booms of uniform section throughout, the compression boom receives increments of stress, not at isolated points but continuously, between the ends and centre, as represented in (*b*), the intensity of these being, at any point similar to the intensity of horizontal shear and varying from the ends towards the centre, as in the diagram (*c*). Diagram (*d*) represents the curve of side bend-

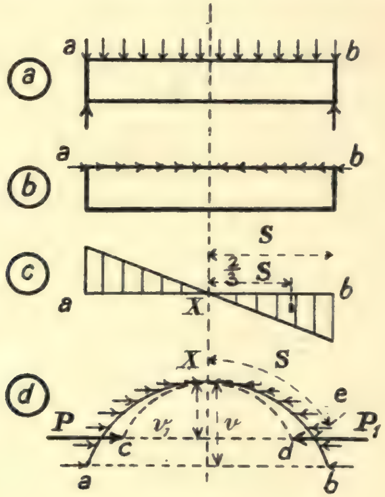


FIG. 87.

ing of the boom. It is in the condition of a strut of Type 2, but instead of carrying single loads applied at its ends, it carries loads equal to P and P_1 distributed over each half length as indicated, the intensity being greater towards the ends.

A section of the strut at the centre X carries a direct load = P equal to the sum of the distributed load, and a bending moment $P \cdot v_1$; v_1 being the distance from the axis at X to the line of action of P the resultant force. The section at X is then in a condition similar to that of the centre of a strut of Type 2, of length *c* X *d* carrying a load P and its reaction P_1 at the ends. In diagram (*c*), the resultant of the forces between the centre and one end is at a distance of $\frac{2}{3}$ S from the centre. In (*d*), owing to the curved form, the resultant P_1 will cut the curve X *b* at a distance from the centre greater than $\frac{2}{3}$ S and the length S_1 of the equivalent strut X *d* is less than X *e* but may be taken as approximately $\frac{2}{3}$ of X *b* . . . . (177)

We thus find that the compression boom of a plain girder, having no lateral support, if of uniform section throughout, and carrying a uniformly distributed load is, so far as side buckling is concerned, approximately in the condition of a strut of Type 2, carrying a load equal to the total stress on the boom at its centre, the length of the equivalent strut being $\frac{2}{3}$ of the span of the girder.

If the section of the boom is reduced towards the ends of the girder, or if the load is not uniformly distributed, the conditions may vary to almost any extent. Again, under moving loads the stresses and their distribution are continually varying, and the conditions are too varied to be accorded full treatment in any treatise.

In a parallel girder with the section of the boom reduced towards the ends, the general average strength may be taken as having about the same ratio to one having booms of uniform section, as that of the tapered strut, in (175), to the corresponding parallel one, that is as having about one half of the strength . . . . (178)

If the girder is reduced towards the ends, while the booms have uniform sections, as in the parabolic bow-string type, so that the compressive stress is constant throughout the length, the total compressive force is applied at the ends only and there are no intermediate increments. If the boom has no lateral support it is in exactly the same condition as a strut of Type 2, of its full length. The torsional rigidity of the girder may assist to some extent in resisting lateral bending, but such stiffness is generally very small.

The compression boom of a girder may receive lateral support in many ways. If of sufficient depth, a pair of main girders carrying a roadway between them may have their upper booms braced together overhead, and the unsupported lengths reduced to the distance between the bracing connections. If the depth of the main girder was not sufficient to permit overhead bracing, the upper booms might be supported by vertical stiffening members on the webs, connected rigidly at their lower ends to the roadway cross girders, and to the booms at their upper ends. Such support is not so effective as the overhead bracing because the vertical stiffeners can deflect and so also can the cross girders to which they are joined.

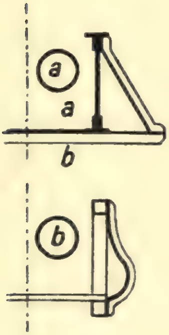


Fig. 88.

In cases of stiffening according to the last-mentioned method it is not uncommon to see girders, such as those of light foot-bridges, with the upper booms stayed as in Fig. 88 (*a*), the stay bar being connected to a bar of comparatively slim proportions running across the bridge to a similar stay to the other girder. Such a connecting bar, if acting, carries bending moments at *a b* and the resistance to angular deflection at that point varies in the manner described in Fig. 56. In some cases of similar bridges with timber girders, so-called stays are to be seen, arranged as in (*b*), a curved bar of rolled angle, or tee, section, or a cast iron bar, being simply attached to the upper and lower booms and not continued across below the bridge floor. Such a stay is quite useless.

When an unsupported girder bends, it may not do so in a single curve to one side, but may assume an S form. In that case it does not necessarily follow that the bending moments are less than with the single curve, because the loads are not applied at the ends, as in the ordinary strut. It is possible, and probable, that under certain loading conditions the moments may be greater with the S curve, as then many parts of the distributed force, or the increments of force, form couples with those increments corresponding to them on the other side of the straight line joining the ends. As in Fig. 89 (*a*), if the deflection curve takes the form of a single arc, the maximum bending moment at X is the sum of the product of all the increments p multiplied by their respective distances from X If, as in (*b*), the curve takes the S form, the maximum bending moment is at X_1 and it depends upon the distribution of p , q , q_1 and p_1 and upon the form of the curve, whether the net result is greater or less than in the case of the single arc.

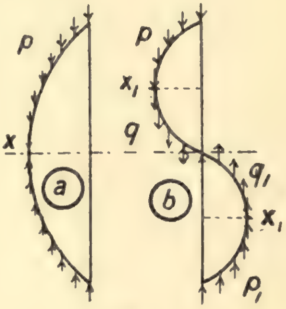


FIG. 89.

Loads considerably off the Axis.

In Chapter VII. we considered the effect of a load on a strut being applied parallel to the axis but at a short distance from it. If the departure is of a considerable amount the problem may still be solved if the exact equations to the curves of bending and of moments are used, but the approximate equations may not give results with sufficient accuracy.

Cases frequently occur in practice where brackets are attached to columns and carry loads at their outer ends, and these generally require different treatment.

Let Fig. 90 (*a*) represent a column, say, between the floor and roof of a factory, having a bracket attached and carrying a load such as a line shaft for machinery, and fixed so that its ends are free to deflect angularly. If the column carries

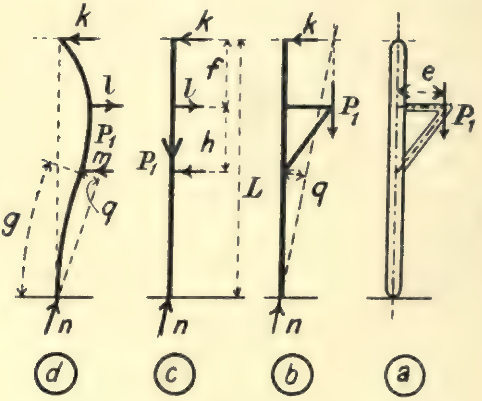


FIG. 90.

no direct load at its upper end but merely serves to carry the bracket and its load, the reactions at the upper and lower ends of the column are as in (*b*). At the upper end the horizontal reaction $k = \frac{P_1 \cdot e}{L}$ At the lower end the inclined reaction n is the balancing force to P_1 and k combined. Its direction is given by a line passing through the lower end and through the intersection of P_1 and k The portion of the column above the bracket is clearly in the condition of a cantilever of the length f with load k at its extreme end—and the bending moment at l is $k \cdot f$ The lower portion below m (as in (*d*)) is in the condition of a portion of a strut of Type I, with a load n applied at its free end, and the portion of the curve can be determined, as the load, section and length g are known, and the ordinate q can be found. It is easier, however, to resolve the force P_1 into a component l (horizontal in this case), and an inclined component along the lower member of the bracket, which, in turn, where it intersects the axis of the strut, is resolved into a component, again equal to P_1, along the axis and one, m, normal to it, as in Fig. 90 (*c*). Then at l we have practically, as before, a moment $= k \cdot f$ and at m a moment $= k\,(f + h) - l \cdot h$ and a direct force P_1, the resulting maximum stress intensity being a combination of those due to the direct force and the bending moment.

If the column, Fig. 90, carries a direct load at its upper end in addition to the load on the bracket, the problem becomes very intricate. In practice, it will nearly always be sufficient to calculate the stresses in the column due to the direct load at its upper end by the ordinary equations herein, and combine the results with the stresses caused by the load on the bracket, calculated as just described.

If the load P_1 on the bracket is not vertical but inclined, as it might be, due to the tension of a driving belt or other cause, the procedure will be just the same. If the inclined force acts in a direction away from the column, the bending moments in the upper part will be increased, and those in the lower part reduced or even reversed. If the inclined force acts towards the column, the stresses will vary in the opposite sense.

If the lower end of the strut is attached so that its axis is fixed in direction, but the upper end free, the problem is modified. The inclined reaction n will not pass through the lower end but through a point on the axis higher up, where there will be a point of contrary flexure, as in Fig. 91 (*a*), and the reaction k will also be different in amount. In this case the easiest method will probably be to first calculate the effect of P_1 if the column was free to deflect at its upper end. It would then assume a curve, as shown by the strong line in (*b*). The lower part of the strut would then be in the condition of part of a strut of Type 1 and of length $X\,l$

carrying load P_1 applied at a distance e from the axis at l The length X l then corresponds to s_z and e to z in equations (137) and (145). From this, v can be found and then the moment at the lower end $= P_1 \cdot v$ and at other points $P_1 \cdot x$ From the slope of the curve at l ($l\,k$ remaining straight), the distance w can be determined. It is then necessary to calculate the force k required to deflect the upper end of the column through the distance w the column acting as a cantilever fixed at its lower end, and the load k applied horizontally at its upper end.

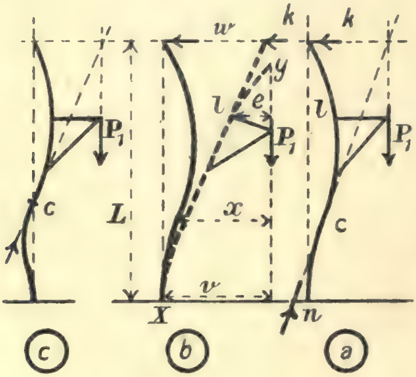


FIG. 91.

At the lower end the resulting bending moment is $P_1 \cdot v - k \cdot L$ Similarly, at any point x below l the moment $= P_1 \cdot x - k \times$ (its distance above x)   The stresses due to these moments below l have to be combined with the stresses due to the direct action of P_1 Above l the bending moment at any point is simply $k \times$ (its distance above that point) and P_1 has no effect directly. Here, again, if the column carries a direct load at its upper end the stresses due thereto must be added to those due to the load P_1 If the load P_1 is comparatively great, the point c of contrary flexure may even move to the opposite side of the original position of the axis, as in (c).

CHAPTER XII.

THE FAILURE OF THE QUEBEC BRIDGE IN 1907.

Description.

THE collapse, in August 1907, of the large steel cantilever bridge which was being constructed over the river St. Lawrence, at Quebec, although in itself regrettable, affords an object lesson on the importance of correct design of struts, much too valuable to be passed over without close investigation.

A full account of the accident and of all the circumstances connected with the construction of the bridge is published in the "Report of Royal Commission, Quebec Bridge Inquiry," of 1908. Articles also appeared in all the principal engineering journals and magazines immediately after the 29th August, 1907, the date of the accident.

The general elevation of the bridge is shown in Fig. 92. The main span of 1800 feet had a central portion formed of girders 675 feet in length, supported on the ends of cantilevers of 562·5 feet in length, extending from the main supporting piers. These cantilevers were connected to other cantilevers extending towards the banks of the river and anchored at their points. The bridge was designed to carry two railway and trolley tracks, two roadways and footways.

The intensities of working stresses allowed for in the design were 24,000 lbs. per square inch in compression, and 18,000 lbs. per square inch in shear. The adoption of these intensities, although higher than would be considered by most engineers to be advisable for a structure of this kind, was not the prime cause of the failure, as investigation shows that at the time of the accident the stress on the member which first failed was only about 0·7 of its final working stress, and corresponded to a direct stress intensity of about 18,000 lb., or 8 tons, per square inch in compression.

At the time of the accident the portions of the structure indicated by the strong lines in Fig. 92 were practically completed. The portions shown in dotted lines were not erected.

Chord A·9·L.

The member which first yielded was that marked A·9·L in the figure, being one chord of the main compression member of the shore cantilever on one side of the bridge. Including all erection appliances and tackle carried by the structure at the time of the failure, it is calculated that the direct thrust upon this member was 14,000,000 lb. The investigation of the Royal Commission shows that, on the 27th August, 1907, two days before the collapse, this member had buckled sideways to the extent of 2·25 inches; that about the middle of August the extent of the buckling was about 0·75 inch; while about the end of July it appears to have been about 0·5 inch; and before that time there was evidently some deflection, but no reliable evidence to show the amount. These particulars enable us to make the necessary investigation.

The construction of chord A·9·L is shown in Fig. 93. The section was uniform throughout and details are shown in Fig. 94. It had four main webs stiffened along each edge by angle bars, and connected together by two double systems of diagonal and transverse bracing bars. The chords were continuous through the points of attachment with other members of the structure and the portion A·9·L was 684 inches in length. At one point in its length there was a main joint and in one bay the bracings were there replaced by plates, but that fact had practically no effect upon the strength as a whole. The dimensions of the principal parts are given in the figure.

The evidence shows that this strut deflected sideways and failed in that direction. In its working position it lay at an inclination longitudinally; the webs were vertical and the bracing bars horizontal so that the deflection was in the plane of these bracings, and the effect of bending will be considered in that direction. The gross sectional area of the chord was 780 square inches and the moment of inertia about the vertical axis line Y Y was about 311,100 in.^{4}, and the radius of gyration 19·97 in. This moment of inertia and radius of gyration, as calculated, are slightly higher than given in the reports referred to.

Stresses in Chord A·9·L.

We have first to determine the Type No. for which the calculations are to be made and a reference to Fig. 92 shows that, at its ends, chord A·9·L had no support in a horizontal plane beyond that afforded by the stiffness of the light transverse and diagonal bracing members connecting together the two main booms of the two cantilevers, combined with what torsional stiffness there was in the other members meeting at its ends. This torsional stiffness is negligible, and as the diagonal and horizontal members referred to would tend to become slack or themselves deflect under load, in the same manner as described in Fig. 56, they would help very little, if at all, especially as they were very light in comparison with A·9·L. Practically, that chord was of Type 2, but to make every allowance for the purpose of this investigation we may adopt, say, Type 2·2, whence

$$s_z = \frac{684}{2\cdot 2} = 311 \text{ in.} \quad . \quad . \quad (179)$$

As the evidence gives fairly reliable data as to the amount of actual deflection, etc., at the time of the collapse, let us use the equations for ideal conditions of loading and calculate the stresses due to bending separately.

Using the constant for Type 2·2, and taking E as 30,000,000, we have, by equations (127) and (110),

$$\mathrm{L}_{c} = 2\cdot 2 \times \mathrm{S} = 2\cdot 2\sqrt{\frac{\mathrm{E\,I_0}}{\mathrm{P}}} \cdot \frac{\pi}{2}$$

$$= 2\cdot 2\sqrt{\frac{30{,}000{,}000 \times 311{,}100}{14{,}000{,}000}} \times \frac{\pi}{2} = 893 \text{ in.}$$

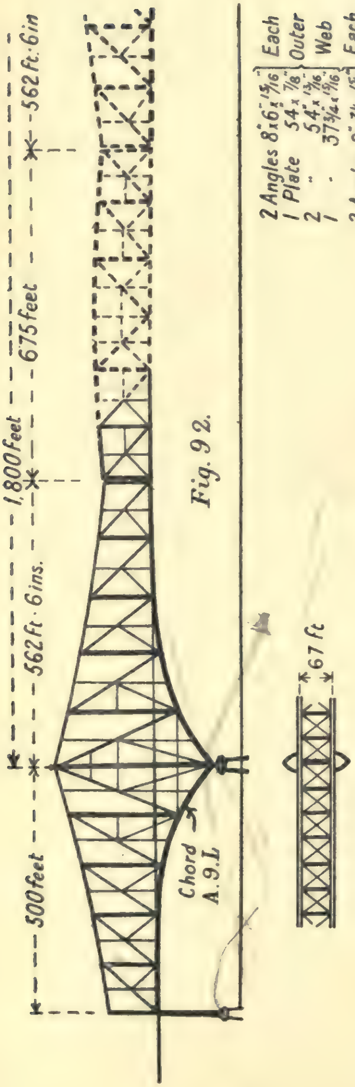


Fig. 92.

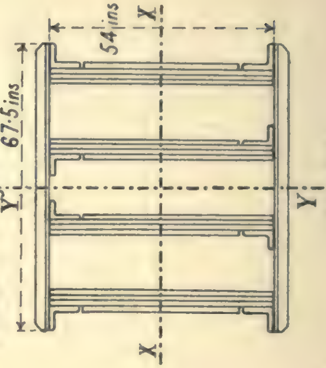


Fig. 94.

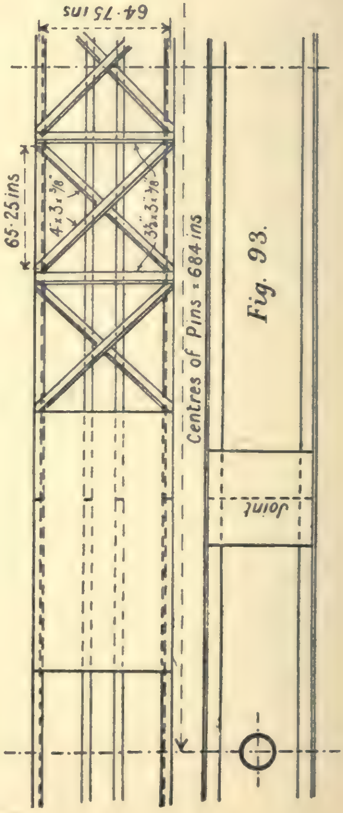
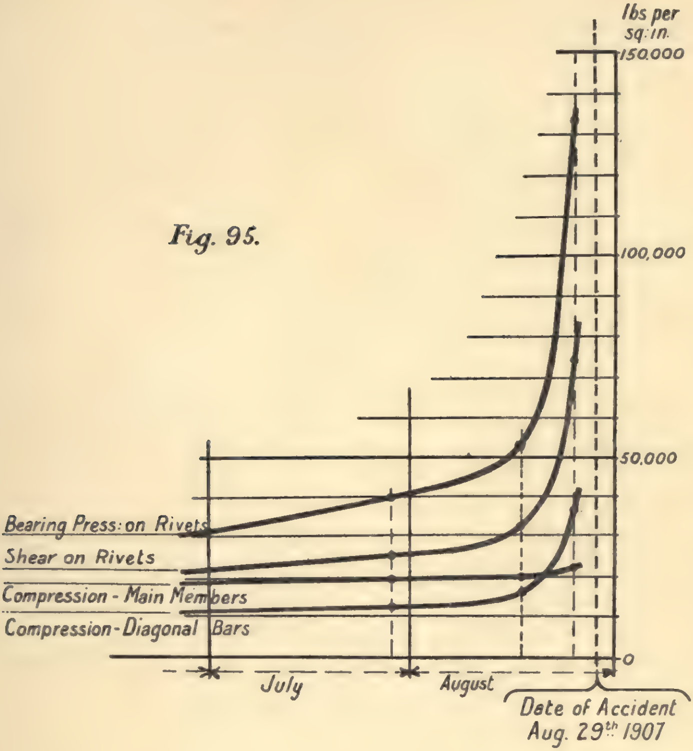


Fig. 93.

Fig. 95.


lbs per sq: in.
150,000
100,000
50,000
0
Bearing Press: on Rivets
Shear on Rivets
Compression - Main Members
Compression - Diagonal Bars
July
August
Date of Accident Aug. 29th 1907

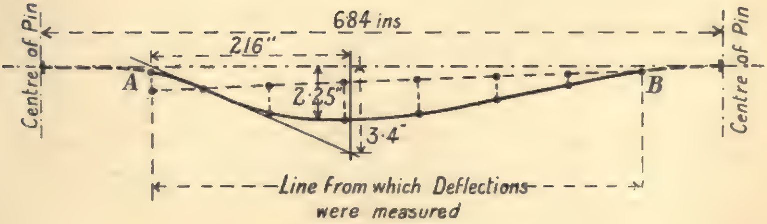

684 ins
216"
2·25"
3·4"
A
B
Centre of Pin
Centre of Pin
Line from which Deflections were measured

Fig. 96.

that is, the strut would not deflect under ideal conditions if less than 893 inches in length. As the actual length was 684 in., there would be no bending stress under ideal conditions and the intensity of the direct stress would be

$$\frac{P}{A} = \frac{14{,}000{,}000}{780} = 17{,}950 \text{ lb. per sq. in.}$$

but this was slightly reduced by the action of the bracing bars, as shown later.

Next consider the stresses in the main members of the strut due to bending action under its own weight. The moment of inertia of the section about the horizontal neutral axis is about 202,200 in.^{4}, and the distance of the extreme fibre from the axis is 27 in. The weight of the chord is not given in the reports, but calculating from the sectional areas of the main members and diagonal bracings, the weight was not less than 165,000 lb. The strut lay at an inclination, the horizontal span being 600 in. It was continuous past the supports at its ends, so that the bending moments were practically those of a continuous beam loaded uniformly. The moment at the centre of the span was therefore

$$M = \frac{165{,}000 \times 600}{24} = 4{,}125{,}000 \text{ in.-lb.} \qquad (180)$$

The maximum fibre stress intensity was

$$\frac{M\,\delta}{I_0} = \frac{4{,}125{,}000 \times 27}{202{,}200} = 550 \text{ lb. per sq. in.} \qquad (181)$$

As the lattice bracing systems lay horizontally, bending of the strut under its own weight produced no stresses in the bracing bars.

Next consider the effect of lateral deflection upon the main members.

When there is no deflection there are no bending stresses in the main members . . . . (182)

With 0·5 in. deflection, the bending moment is

$$M = 14{,}000{,}000 \times \cdot 5 = 7{,}000{,}000 \text{ in.-lb.}$$

The maximum fibre stress intensity $= \dfrac{M\,\delta}{I_0}$

$$= \frac{7{,}000{,}000 \times 33\cdot 75}{311{,}100} = 760 \text{ lb. per sq. in.} \qquad (183)$$

With 0·75 in. deflection

$$M = 14{,}000{,}000 \times \cdot 75 = 10{,}500{,}000 \text{ in.-lb.}$$

and the maximum fibre stress intensity

$$= \frac{10{,}500{,}000 \times 33\cdot 75}{311{,}100} = 1140 \text{ lb. per sq. in.} \qquad (184)$$

With 2·25 in. deflection

$$M = 14{,}000{,}000 \times 2\cdot 25 = 31{,}500{,}000 \text{ in.-lb.}$$

and the maximum fibre stress intensity

$$= \frac{31{,}500{,}000 \times 33\cdot 75}{311{,}100} = 3410 \text{ lb. per sq. in.} \qquad (185)$$

Stresses in Bracing Bars.

If we examine the system of bracing adopted we find it to be that described in Fig. 82, and the bracing bars carried some of the direct thrust. The diagonal bars have a sectional area of 2·48 sq. in. each and the transverse bars 2·3 sq. in. each, therefore, taking the strut as a whole, there being two double systems of bracing, the diagonal bars have a total area of $2\cdot 48 \times 2 = 4\cdot 96$ sq. in., and the transverse bars $2\cdot 3 \times 2 = 4\cdot 6$ sq. in. The angle ψ between the diagonal and the axis of the strut is almost 45°, giving $\sin\psi = 0\cdot 707$, and $\cos\psi = 0\cdot 707$. The fact that the whole of the metal in the section is not concentrated into two simple booms connected by the bracings does not affect the proportion of the direct load carried by the booms and bracing systems, because, as previously explained, the booms and bracing systems share the load between them simply in the ratio in which they resist being shortened under load.

Using equations (172), (173) and (174), we have

$$\tfrac{1}{2}\,\mathrm{P} \;=\; \frac{14{,}000{,}000}{2} \;=\; 7{,}000{,}000 \text{ lb. ; and}$$

$$\alpha \;=\; \frac{780}{2} \;=\; 390 \text{ sq. in.}$$

Stress in diagonal β

$$= \frac{\tfrac{1}{2}\,\mathrm{P}}{\dfrac{\alpha}{\cos^2\psi}\left(\dfrac{1}{\beta} + \dfrac{2 \times \sin^3\psi}{\gamma}\right) + \cos\psi}$$

?

$$= \frac{7{,}000{,}000}{\dfrac{390}{0{\cdot}707^2}\left(\dfrac{1}{4{\cdot}96} + \dfrac{2 \times 0{\cdot}707^3}{4{\cdot}6}\right) + 0{\cdot}707} \;=\; 25{,}200 \text{ lb.}$$

Stress in transverse members γ

$$= \;2 \times (\text{stress in } \beta) \times \sin\psi$$

$$= \;2 \times 25{,}200 \times 0{\cdot}707 \;=\; 35{,}600 \text{ lb.}$$

and stress in α

$$= \;\tfrac{1}{2}\,\mathrm{P} - (\text{stress in } \beta) \times \cos\psi$$

$$= \;7{,}000{,}000 - 25{,}200 \times 0{\cdot}707 \;=\; 6{,}982{,}000 \text{ lb.}$$

From these we have, due to the direct action of P:—
Total stress in one diagonal angle bar

$$= \frac{25{,}200}{2} \;=\; 12{,}600 \text{ lb.} \qquad . \qquad . \quad (186)$$

Total stress in one transverse angle bar

$$= \frac{35{,}600}{2} \;=\; 17{,}800 \text{ lb.} \qquad . \qquad . \quad (187)$$

Total stress in main members of strut

$$6{,}982{,}000 \times 2 \quad = \quad 13{,}964{,}000 \text{ lb.} \qquad . \quad (188)$$

the intensity of the stress in the main members being

$$\frac{13{,}964{,}000}{780} \quad = \quad 17{,}900 \text{ lb. per sq. in.} \qquad . \quad (189)$$

against 17,950 lb. per sq. in. if the bracing systems had not taken any of the direct load.

It may be argued that because the wing of one of the diagonal angles, as shown in Fig. 93, was cut away to allow the other to pass, the whole sectional area of the angles should not be used in the calculations. It was shown, however, in connection with equation (172), etc., that the stresses in these bars, due to the direct load, depend entirely upon the extent of shortening under load and the corresponding elastic reactions, and as the cutting away only extended for a very short portion of the whole length, the result would not be affected to an appreciable extent, especially as the riveted connections between the angles and the booms gave extra stiffness which would tend to increase the stresses quite as much as the cutting away would reduce them.

From the report of the Royal Commission it appears that in the original calculations for this strut, owing to the fact that one of the wings of the diagonal angle bar was thus cut away, it was *considered* that only the area of the remaining wing was acting, and the sectional areas were calculated accordingly. It will be seen that as regards the stresses now under consideration, this view is quite erroneous. Where stresses depend on elastic reactions, the whole of the material employed will act, whether it is *considered* to do so or not. This is a case where the addition of more metal to the bracing bars would increase the stresses in them. This can easily be proved by trial with the above equations. The assumption that only part of a section acts would, in such a case as this, be quite sufficient to lead to the collapse of the structure, because if only half of the section was considered and connecting rivets put in accordingly, it would be found that the real stress on the diagonal bar was nearly double what it was supposed to be, and the stresses on the rivets nearly double the amount provided for. This would actually have been the case in connection with chord A·9·L had it not been that more rivet area was provided than the erroneous calculations indicated. Even then, however, the area was not sufficient owing to the direct stresses being neglected.

It is interesting to note, that some measurements were taken of the lateral deflection of chord A·9·L two days before the accident. From these measurements the deflection curve in Fig. 96 is drawn. The measurements were taken from a straight line AB, at the points indicated. That line does not extend from end to end of the strut, but between two intermediate points, and the total deflection from the axis is not given. The portion of the curve shown by a full line extends as far as the measurements were taken. The measurements for this curve are quite definite in a longitudinal direction, but there is nothing to show the correct positions of the points relatively to the axis in a transverse direction. In the figure the curve has been completed by supposition, as shown by the dotted portions, and this has been done in the most favourable way possible. This gives the maximum deflection from the axis as 2·25 in. The report of the Commission states that there was an average deflection of 1·75 in. in the four webs, but that was in a length of about 40 feet, and not in the whole length of the strut. So far as concerns the stresses in the bracing bars, due to deflection, whether 1·75 in. or 2·25 in. was the amount, will not affect the stress of 77,800 lb. given below, as it is arrived at from the measured slope of the curve, and not from the amount of the deflection. If we took 1·75 in. as the maximum instead of 2·25 in., the stresses of 17,300 lb. and 25,950 lb. given below for the smaller deflections would be much increased, and not reduced, and therefore, taking 2·25 in. ensures that they are not overstated. On the other hand taking the figure of 1·75 in.

4760

instead of 2·25 in. would reduce the stress intensity of 760 lb. given for the main section due to bending under thrust, but that would not appreciably affect the final result. On the whole, therefore, the case against the strut has been put as mildly as possible. Although the curve is somewhat irregular, the steepest slope is nearly the same as that allowed for in equation (165), for a similar rate of deflection. It is fortunate that these measurements are available, as they enable the stresses to be calculated very closely, and all doubt as to the cause of the failure is removed.

We have next to find the approximate stresses in the diagonal bars due to the bending or deflection. From Fig. 96 the slope of the actual curve into which the strut was bent was, at its steepest part, at the rate of 3·4 in. in a length of 216 in., giving an angle i having

$$\sin i = 0{\cdot}0157.$$

Then, by equation (158)

$$R = P \times \sin i = 14{,}000{,}000 \times 0{\cdot}0157 = 220{,}000 \text{ lb.}$$

Then the total stress on the diagonals $= \dfrac{R}{\sin \psi}$, and as there were, in all, four sets of diagonals, the stress on each diagonal bar

$$= \frac{R}{4 \cdot \sin \psi} = \frac{220{,}000}{4 \times 0{\cdot}707} = 77{,}800 \text{ lb.} \qquad (190)$$

the total deflection being 2·25 in.

If we assume that, when the deflection was less, the curvature was of a similar form, we have approximately, the stress on one diagonal bar with total deflection of 0·5 in.

$$\frac{77{,}800 \times 0{\cdot}5}{2{\cdot}25} = 17{,}300 \text{ lb.} \quad . \quad (191)$$

and with a deflection of 0·75 in.

$$\frac{77{,}800 \times 0{\cdot}75}{2{\cdot}25} = 25{,}950 \text{ lb.} \qquad . \quad (192)$$

These stresses appear to be very high, but there is no doubt about their value as they follow from the measurements reported, and the deflection of 2·25 in. is a very large amount for such a strut.

Stresses on Rivets.

Consider now the stresses on the rivets in the ends of the diagonal bars. These rivets were 0·875 in. in diameter, and there were two rivets in each bar. The maximum stress on each diagonal bar was the sum of the stresses due to the direct thrust and the bending action, as shown in the table opposite. The corresponding intensity of the shearing stress on the rivets, assuming uniform distribution is

With no deflection

$$= \frac{12{,}600}{2 \times 0{\cdot}6 \text{ sq. in.}} = 10{,}500 \text{ lb. per sq. in.} \quad (193)$$

With 0·5 in. deflection

$$= \frac{29{,}900}{2 \times 0{\cdot}6} = 25{,}000 \text{ lb. per sq. in.} \quad . \quad (194)$$

With 0·75 in. deflection

$$= \frac{38{,}550}{2 \times 0{\cdot}6} = 32{,}120 \text{ lb. per sq. in.} \quad . \quad (195)$$

With 2·25 in. deflection

$$= \frac{90{,}400}{2 \times 0{\cdot}6} = 75{,}350 \text{ lb. per sq. in.} \quad . \quad (196)$$

Each rivet passed through the wing of the angle bar which was 0·375 in. in thickness, giving a bearing area of (0·875 × 0·375) sq. in. The intensity of the bearing pressure, assuming uniform distribution is, therefore

With no deflection

$$= \frac{12{,}600}{2 \times 0{\cdot}875 \times 0{\cdot}375} = 18{,}800 \text{ lb. per sq. in.} \quad (197)$$

With 0·5 in. deflection

$$= \frac{29{,}900}{2 \times 0{\cdot}875 \times 0{\cdot}375} = 44{,}600 \text{ lb. per sq. in.} \quad (198)$$

With 0·75 in. deflection

$$= \frac{38{,}550}{2 \times 0{\cdot}875 \times 0{\cdot}375} = 57{,}500 \text{ lb. per sq. in.} \quad (199)$$

and with 2·25 in. deflection

$$= \frac{90{,}400}{2 \times 0{\cdot}875 \times 0{\cdot}375} = 134{,}900 \text{ lb. per sq. in.} \quad (200)$$

The results of these calculations are summarised in the following table. In arriving at these stresses nothing has been allowed for wind pressure, or other causes of temporary stress.

SUMMARY OF STRESSES.

IN EXTREME FIBRE OF SECTION OF CHORD A·9·L (CENTRE SECTION).

		STRESS INTENSITY.			
Deflection		Nil	0·5 in.	0·75 in.	2·25 in.
Cause of Stress					
Direct thrust		17,900	17,900	17,900	17,900
Bending under thrust		..	760	1,140	3,410
Bending under own weight		550	550	550	550
Totals	lbs. per sq. in.	18,450	19,210	19,590	21,860
	tons per sq. in.	8·25	8·6	8·75	9·75

STRESS IN EACH DIAGONAL ANGLE BAR.

Deflection		Nil	0·5 in.	0·75 in.	2·25 in.
Cause of Stress					
Direct thrust		12,600	12,600	12,600	12,600
Bending under thrust . .		..	17,300	25,950	77,800
Total on each bar. lbs. . .		12,600	29,900	38,550	90,400
,, ,, lbs. per sq. in.		5,080	12,000	15,530	36,450
,, ,, tons per sq. in.		2·25	5·35	6·95	16·3
Shearing stress per sq. in. on 0·875 in. rivets	lbs.	10,500	25,000	32,120	75,350
	tons	4·7	11·2	14·35	33·6
Bearing pressure on rivets per sq. in.	lbs.	18,800	44,600	57,500	134,900
	tons	8·4	20·0	25·7	60·0

Increase of Stress before Collapse.

From these results and the dates on which the various deflections were reported, the diagram, Fig. 95, has been made. The four curved lines show how, during several weeks before the final collapse, the stress intensities were increasing.

The intensity of the compressive stress in the main members of the strut, although above the usual average, was considerably below the limit of elasticity and therefore not, in itself, dangerous.

The stress intensity in the diagonal members, taken over the gross area, was at first less than that in the main members, but it afterwards increased rapidly, as the deflection increased.

The shearing stress intensity on the rivets was high, even if there had been no deflection, and with a very moderate deflection, dangerously high. Having regard to the excessive bearing pressure on the same rivets, it is apparent that they would distort and give way. The rivets must have given way, to some extent, before the final collapse. The diagram indicates that they were distorting and allowing the deflection to increase for several weeks before the final failure.

The deflection curve in Fig. 96 appears, at first sight, to be that of a continuous strut, but the attachments at the ends could not afford such support. If observations had been made it would certainly have been discovered that the bracing bar rivets were distorting and allowing the bars to slide, and that such was the cause of the deflection. In any case the question of continuity does not affect the above calculations, as the direct stress intensities are those for ideal conditions and no deflection, while the bending stresses are calculated from the measured deflections.

These rivets could not stand up against such stresses, and it is not suggested that they were actually resisting these stresses at the various dates. What undoubtedly happened was that at least three or four weeks before the accident, owing to the high shearing and bearing pressures, the rivets were becoming distorted, and the bracing bars, being allowed to slide at their attachments, were relieved of part of the stresses due to the direct thrust on the strut. The whole strut then deflected owing to the bracings becoming slacker. Next, owing to the increased deflection, the shearing and bearing pressures were increased and further distortion of the rivets allowed still further deflection, and so on. In addition, owing to weight being gradually added to the bridge, and to some of the erecting cranes being moved further out on the river cantilever the day before the collapse, the stresses became so high that some of the rivets and bracing bar ends gave way completely, and the general collapse followed immediately. The rivets in the ends of the bracing bars were so few in number—there being 4 sets of bars with only 2 rivets each—that the whole bridge depended upon 8 rivets, stressed so highly that the failure of one of them would almost certainly be followed by that of the whole structure.

An examination of the various drawings and photographs of parts of the wreckage shows conclusively that these bracing bar ends and rivets had nearly all bent or sheared. After the accident large scale models of chord A·9·L were made and tested. Owing to the defective design, the results of these tests are of no value whatever in connection with the theories of stresses, etc., but they confirm the above conclusions as they all failed in the same manner, the rivets or ends of the bracing bars being the first to fail in all cases.

Cause of Failure.

This investigation shows that the real cause of the failure of the bridge was the omission to provide for the stresses in the bracing bars and connections, due to the direct thrust on the strut.

These stresses, added to those due to a moderate amount of normal bending, were sufficient to cause the strut to commence to deflect, and even if the rivets then gave way sufficiently to relieve the bracing bars of the whole of the direct stresses, the increase of the stresses due to the bending, which had been brought about, was alone sufficient to complete the collapse. When the deflection reached the amount of 2·25 in. the stress in one diagonal bar, due to bending alone, was 77,800 lb., causing a shearing stress intensity of 64,800 lb., or 29 tons, per square in., and a bearing pressure intensity of 116,100 lb., or 52 tons, per square in. on the rivets.

The stresses on the bracing bars due to normal bending, etc., were also underestimated, as it was assumed, in the calculations which were made, that the axis of the strut was fixed at both ends as in Type 4, but this, in itself, led to no danger, because the rivets and bars could not be made, in practice, so small as indicated and they were made greater.

The failure was not due to defects in the usual theories made use of in designing struts. Although most of such theories are not absolute, the errors caused by their use could not, in themselves, be so great as to overbalance the factors of safety generally used.

In this case it so happened that, by chance, the sectional area of the diagonal bracing bars was made greater than the calculations which were made indicated as being necessary, and if the rivets securing their ends had been increased in number so as to have been capable of utilising the whole strength of these bars, it is certain that the bridge would now have been complete and carrying the traffic as intended, and probably nothing would have been suspected about its safety, although the margin would have been small. Having regard to the fact that no secondary stresses were allowed for in the design, the stress intensities adopted were undoubtedly high, for the material used, and will probably never be repeated in future.

If the transverse members of the bracing systems had been entirely removed, so as to prevent stresses due to the direct thrust, and if the bracing of the two inner members of the struts had been made complete, the bridge would have stood and had a small margin of safety.

From the available data it appears that the stress intensities in chord A·9·L were higher than in the adjoining members. Some of the plates near the main joint were not riveted up until some time after erection, and shortly before the accident some of these plates had been removed for inspection, and replaced. It is possible that the general strength may have been temporarily reduced by such operations, although there were no signs of undue local bending near the joint after the event.

The fact that the strut bent in the direction of its greatest moment of inertia, and not even approximately in the direction in which its stiffness should have been least, shows that such bending was not due to real elastic deflection, but can only be accounted for by other causes, such as a partial shearing or deformation of rivets, etc.

Details in the Design.

It is instructive to examine a number of details in the design of chord A·9·L in illustration of some points in design already referred to herein.

The section of the strut (Fig. 94) relatively to the axis Y Y has stiffening angles with wings *a* and *b* of comparatively slim proportions projecting beyond the main parts of the section, as referred to in (37).

The moment of inertia I_0 of the whole section about Y Y has been given already, as 311,100 in.^{4} The sectional area is 780 in.^{2}, and δ is 33·75 in. Then the moment of resistance to bending, is, for unit stress intensity,

$$M\,r = \frac{I_0}{\delta} = \frac{311{,}100}{33 \cdot 75} = 9220 \text{ in.}^3 \quad . \quad (201)$$

the radius of gyration is

$$r = \sqrt{\frac{311{,}100}{780}} = 19 \cdot 97 \text{ in.} \quad . \quad (202)$$

and the ratio $\frac{L}{r} = \frac{684}{19 \cdot 97} = \frac{34 \cdot 3}{1} \quad . \quad . \quad (203)$

Now, by the method of (37), etc., if the wings *a* and *b* of these angles were entirely removed the sectional area would be reduced to 761 in.^{2}; the corresponding moment of inertia I_0 being 293,730 in.^{4}; and δ = 28·6875 in. Then we should have

$$M\,r = \frac{I_0}{\delta} = \frac{293{,}730}{28 \cdot 6875} = 10{,}230 \text{ in.}^3 \quad . \quad (204)$$

the radius of gyration

$$r = \sqrt{\frac{293{,}730}{761}} = 19 \cdot 65 \text{ in.} \quad . \quad (205)$$

and the ratio

$$\frac{L}{r} = \frac{684}{19 \cdot 65} = \frac{34 \cdot 8}{1} \quad . \quad . \quad (206)$$

By the criterion of (37), the breadth of the rectangle having δ the same as that of the section without the wings of the angles (and therefore 28·6875), and having the same I_0, is

$$B = \frac{3\,I_0}{2\,\delta^3} = \frac{3 \times 293{,}730}{2 \times 28 \cdot 6875^3} = 18 \cdot 6 \text{ in.} \quad . \quad (207)$$

If the small rectangles equivalent to the angle wings are added, their breadth of 1·875 in. is $\frac{1\cdot875}{18\cdot6}$ = about $\frac{1}{10}$ of the breadth, B, and therefore much less than the limit of $\frac{1}{3}$ given by equation (35). It will be seen that by removing the wings of these angles the resistance to bending would be increased in the ratio $\frac{10{,}230}{9{,}220}$ or nearly 11 per cent. The resistance of the portions of the boom between the points of attachment of the bracings would, however, be much reduced, and the section without the angle wings would be quite unsuitable for a strut.

If, instead of removing the wings of the angles, the angles were placed on the inner sides of the webs, instead of the outer sides, the total sectional area would remain at 780 in.^{2} The moment of inertia becomes

$$I_0 = 303{,}580 \text{ in.}^4$$

and

$$\delta = 28\cdot6875 \text{ in.}$$

Then

$$M\, r = \frac{303{,}580}{28\cdot6875} = 10{,}580 \text{ in.}^3 \qquad . \quad (208)$$

The radius of gyration

$$r = \sqrt{\frac{303{,}580}{780}} = 19\cdot72 \text{ in.} \qquad . \quad (209)$$

and the ratio

$$\frac{L}{r} = \frac{684}{19\cdot72} = \frac{34\cdot6}{1} \qquad . \quad . \quad (210)$$

Comparing these figures with those for the original section, we see that the ratio $\frac{L}{r}$ is altered very slightly and the sectional area remains the same, while the resistance to bending about the axis Y Y is greater in the ratio of $\frac{10{,}580}{9220}$ or 14·75 per cent. If we compare the original strut with one having these angles turned inwards, we have, for Type 1, for the original section,

$$P = 39{,}500{,}000 \text{ lb.}; \; p' = 50{,}640$$
$$\text{and } p'' = 9360 \text{ lb. per sq. in.}$$

By (145),

$$S = \sqrt{\frac{E\, I_0}{P}}\left(\frac{\pi}{2} - c \cdot \frac{\delta}{r} \cdot \frac{p'}{p''}\right)$$

$$= \sqrt{\frac{30{,}000{,}000 \times 311{,}100}{39{,}500{,}000}}\left(\frac{\pi}{2} - \frac{33{\cdot}15 \times 50{,}640}{10 \times 19{\cdot}97 \times 9360}\right)$$

$$= 311 \text{ in.}$$

For the altered section,

$$P = 40{,}400{,}000 \text{ lb.}; \; p' = 51{,}800$$
$$\text{and } p'' = 8200 \text{ lb. per sq. in.}$$

and

$$S = \sqrt{\frac{30{,}000{,}000 \times 303{,}580}{40{,}400{,}000}}\left(\frac{\pi}{2} - \frac{28{\cdot}65 \times 51{,}800}{10 \times 19{\cdot}72 \times 8200}\right)$$
$$= 311 \text{ in.}$$

In both cases 311 × the Type No. 2·2 = 684 = the length of chord A·9·L in the bridge. Therefore if the angles in question had been turned inwards the strength would have been increased in the ratio of $\frac{40{,}400{,}000}{39{,}500{,}000} = 2 \cdot 3$ per cent.; or, for the same strength, the sectional area could have been reduced to about that extent, and if all the struts in the bridge could have been improved in proportion the saving in cost would have been great. Another advantage of turning these angles inwards would have been that they would be less liable to buckle locally, in the manner described in (154), as the intensity of stress at their extreme edges would be much reduced. It appears, from photographs, etc., as if some of these angles did buckle at the time of the accident.

In the foregoing calculations no allowances have been made for local flexure. Of the four portions of the main section of the strut the outer ones were as in Fig. 97. The moment of inertia about the axis X X is 354 in.^{4}, and r = 1·347 in.

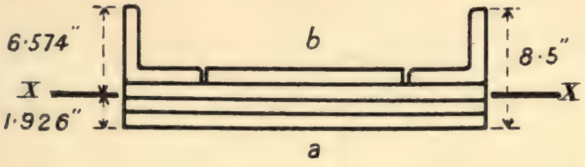


Fig. 97.

The unsupported length between the bracing connections is 72 in. and $\frac{L}{r} = \frac{72}{1 \cdot 347} = \frac{54}{1}$. With that ratio and ideal conditions for Type 2 there should be no local bending between the bracing connections, but as the ratio $\frac{\delta}{r}$, on one side of the axis, is $\frac{6 \cdot 574}{1 \cdot 347} = \frac{5}{1}$, a reference to Fig. 45 will show that under any disturbing causes the efficiency of the section is very low, and the strength of the column as a whole must be considerably reduced.

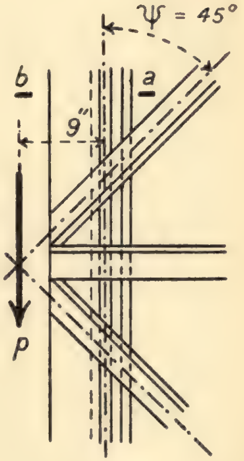


Fig. 98.

Another point to be noticed is that, as in Fig. 98, the diagonal members of the bracing did not intersect on the axis of the outer portion of the main section, but at a point about 9 in. beyond. The result of such a departure is a bending moment in the main member, equal to the longitudinal component of the stresses in the diagonals multiplied by the amount of the departure. From the table, the maximum stress, with a deflection of 0·5 in. only, on a diagonal was

17,300 lb., due to the bending action only. As the strut is braced on both sides, the total component acting on the main member is

$$\begin{aligned} p &= 4 \times (\text{stress on diagonal}) \times \cos \psi \\ &= 4 \times 17{,}300 \times 0{\cdot}707 = 48{,}930 \text{ lb.} \end{aligned}$$

and the bending moment therefore

$$= p \times 9 = 48{,}930 \times 9 = 440{,}370 \text{ in.-lb.}$$

This bending moment would cause a maximum compressive stress intensity on the outer fibre of the section, Fig. 97,

$$= \frac{M\,\delta}{I_0} = \frac{440{,}370 \times 6{\cdot}574}{354} = 8200 \text{ lb. per sq. in.}$$

in the bays where the member bent so that the side marked *b* was on the concave side of the curve of bending.

It will be seen that this stress is not one that should have been neglected. If added to the maximum stress intensity given in the table, the result becomes more serious, having regard to the somewhat high intensities allowed for in the design. Of course these local stresses are affected by the fact of the main members of the strut being continuous and some of the stresses being absorbed in different directions, but in this case the effect of the bad intersection of the bracing bars is very great and in itself sufficient to condemn the design. Wherever the stresses go they are of serious amount.

The method of attaching the bracing bars to the main members has been criticised, and it has been stated that the diagonal members carrying compressive stresses were liable to buckle or deflect, owing to the load being applied at one side and considerably off the axis of the bar. At first sight this would appear to be so, but, in reality, the conditions are not those of a free ended strut, or Type 2. The ends were veted to the main members of the strut and these were y massive in comparison with the bracings. As in Fig. 99, the force P acting at the end of the bracing angle appears to be applied at the extreme edge of the bar, but the attachment was so rigid that it would be almost impossible for the axis of the angle bar to deflect angularly, as in doing so it would have to distort the massive member to which it was connected. In deflecting, the angle bar would have to take a double curvature as indicated by the dotted lines, and the bending moment at the ends would be resisted by the rivets, two in this instance. The stresses in the rivets would cause reactions acting on the rivets in the direction of the arrows t and t_1 forming a couple, having a moment of exactly the amount necessary to transfer the action of the force P laterally until it coincided with the axis of the angle bar. If the main member yielded slightly, P would not be transferred quite on to the axis, but it is improbable that there was any yielding. Therefore these diagonal bars were practically in the condition of struts of Type 4.

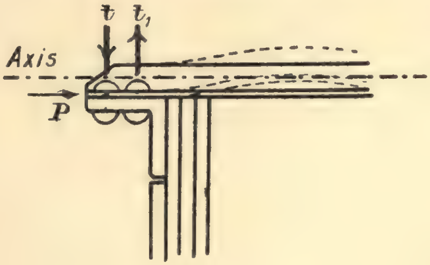


FIG. 99.

It has been pointed out that of the portions of the main members of the chord A·9·L the two inner ones, *a* and *b* in Fig. 100, were imperfectly braced together, and that to enable them to take their share in resisting deflection additional bracings were necessary, such as indicated by dotted lines in the figure. This is quite correct, but these two members were so close to the axis of the strut that, if all other details had been sufficient, the effect of the omission of these bracings alone would not have been great. The two members *a* and *b* being imperfectly braced would, if they had to stand by themselves, be under the same conditions as the strut, described in Fig. 67, with batten plates at intervals and would be very inefficient, but, being combined with the other members which were fully braced, the conditions were very different. The moment of inertia of the section of these two inner members about the central axis is less than 8 per cent. of the whole section. At the same time, the defect in the bracing of the inner members causes local stresses in the whole of the bracing bars.

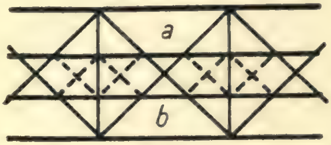


FIG. 100.

CHAPTER XIII.

SUMMARY OF EQUATIONS AND PRACTICAL EXAMPLES OF DESIGN.

THE following are summaries of the principal letters and symbols used throughout this work together with their meanings; and also of the equations required in designing struts. These may save time in referring to the various pages for the information.

Ordinary Letters.

A = Area of section.

B = Breadth of a rectangle, or dimension parallel to the axis.

D = Diameter of any figure, normal to the axis.

E = Modulus of elasticity.

I = Moment of inertia, about any axis.

I_0 = Moment of inertia, about a neutral axis.

I_{0X} = Moment of inertia, about the neutral axis X X and so on.

K = A constant, $= \sqrt{\frac{E\,I_0}{P}}$.

L = Total length of a strut.

M = Bending moment.

O = Origin of co-ordinates.

P = Total load on a strut.

R = Transverse shearing force, at any point.

R_s = Maximum transverse shearing force.

S = Half-length of an elastic curve of single inflexion, or distance from centre point of curve to a point of contrary flexure.

F = Elliptic Function, of first kind.

F_1 = Complete Elliptic Function, of first kind.
E = Elliptic Function, of second kind.
E_1 = Complete Elliptic Function, of second kind.
c = A constant.
f = Maximum stress intensity.
f_1 = Reduced stress intensity, after allowing for local flexure.
i = Angle between axis and tangent to curve of bending, at any point.
i_Y = Angle between axis and tangent at point where curve meets line of action of P or at a point of contrary flexure.
l = Distance between points of intersection of bracing bars, or length of panel.
p' = Stress intensity due to direct load.
f p'' = Stress intensity due to bending action.
r = Radius of gyration.
s = Any distance measured along a curve of bending, from the central point.
s_2 = Distance measured along a curve of bending, from the central point to the end, when the curve does not reach the line of action of P
δ v = Maximum deflection, or greatest distance from the line of action of P to the curve of bending.
x = Any deflection, other than the maximum.
z = An initial departure of the line of action of P from the axis of the strut.

Greek Letters.

$\left.\begin{matrix}\alpha\\ \beta\\ \gamma\end{matrix}\right\}$ = Sectional areas of members of a compound system of bracing, as in Fig. 82.
δ = Distance of the extreme fibre of a section from the neutral axis.
ρ = Radius of curvature.
$\frac{1}{\rho}$ = Curvature.

λ = Angle between any axis of a section and the principal axis.
θ = Angle which is the modulus of the Elliptic Function.
ϕ = Angle which is the amplitude of the Elliptic Function.
ψ = Angle between any bracing bar and the axis of the strut.

Equations for Practical Use.

(2) $\rho = \frac{E\,I_0}{M} = \frac{E\,I_0}{P\,x}$

(19) $r = \sqrt{\frac{I_0}{A}}$

(37) $B = \frac{12\,I_0}{D^3}$

(43) $I_{0x_1} = I_{0X} \cdot \cos^2\lambda + I_{0Y} \cdot \sin^2\lambda$

(48) $v = \frac{p''\,I_0}{P\,\delta}$

(58) v varies inversely as δ

(59) For Isotropic Section worst direction for bending is in direction of greatest radial dimension.

(61) With Isotropic Section bending always takes place in direction of action of deflecting force.

(83) $K^2 = \frac{E\,I_0}{P}$

(84) $K^2 = \rho x = \rho_v v$

(88) $\cos i = 1 - \frac{v^2}{2\,K^2}\left(1 - \frac{x^2}{v^2}\right)$

(101) $\theta = \frac{i_Y}{2}$

(110) $S = \sqrt{\frac{E\,I_0}{P}} \cdot \frac{\pi}{2} = K \cdot \frac{\pi}{2}$

(111) $P = \frac{E\,I_0\,\pi^2}{4\,S^2}$

(127) L = (Type Number) × S
or (Type Number) × s_z

(133) $v = \frac{r^2}{\delta}\left(\frac{f}{p'} - 1\right)$

(134) $v = \frac{I_0}{\delta}\left(\frac{f}{P} - \frac{1}{A}\right)$

(135) $\frac{L}{r} = \sqrt{\frac{E}{p'}} \cdot \pi$ or $p' = \frac{E\,\pi^2}{\left(\frac{L}{r}\right)^2}$

(142) $s_z = \sqrt{\frac{E\,I_0}{P}}\left(\frac{\pi}{2} - \frac{z}{v}\right) = K\left(\frac{\pi}{2} - \frac{z}{v}\right)$

(145) $s_z = \sqrt{\frac{E\,I_0}{P}}\left(\frac{\pi}{2} - c \cdot \frac{\delta}{r} \cdot \frac{p'}{p''}\right)$

$= K\left(\frac{\pi}{2} - c \cdot \frac{\delta}{r} \cdot \frac{p'}{p''}\right)$

(150) $f_1 = \frac{E\,\pi^2}{\left(\frac{l}{r}\right)^2}$

(165) $R_z = 2{\cdot}5\,\frac{P\,v}{S}$

(166) Stress in Bracing Bar $= \frac{R_z}{\sin\psi}$ for single system.

(167) do. do. $= R_z \times \frac{\sqrt{D^2 + l^2}}{D}$

(172) Stress $\beta = \dfrac{\frac{1}{2}P}{\frac{\alpha}{\cos^2\psi}\left(\frac{1}{\beta} + \frac{2\sin^3\psi}{\gamma}\right) + \cos\psi}$

(173) Stress γ $= 2 \cdot \text{stress}\ \beta \cdot \sin \psi$

(174) Stress α $= \frac{1}{2}P - \text{stress}\ \beta \cdot \cos \psi$

(176) For equal lengths, load on parallel strut may be about twice the load on a strut with curved taper, as Fig. 85.

or Strength of curved tapered strut is about half of that of parallel strut.

As explained in a general way in Chapter VII., the process of designing a strut must be, to a great extent, a tentative one.

Take the Rankine-Gordon formula, which has been the most used. It gives

$$P = \frac{f\,A}{1 + a\,\frac{L^2}{r^2}}$$

Here we have factors P L and f which are almost invariably fixed beforehand, and a section has to be designed, having A and r of values which will satisfy the equation. There is also the empirical constant a the value of which has to be taken from the results of experiments, and which covers everything relating to E and to the nature of the supports, etc. As a is not varied for different forms of section, it follows that the same result will be obtained for any section having A and r of same values, and it is possible to design sections of endless variety of form which would comply with these conditions. It is necessary to design a trial section and insert the corresponding values of A and r into the right hand side of the equation. Then, if the resulting value of the right hand side has not the same value as the known value of P the section must be enlarged or reduced and the new values of A and r inserted, and so on until the result is considered near enough.

Now take the equation (145) for Type 1.

$$s_z = \sqrt{\frac{E\,I_0}{P}}\left(\frac{\pi}{2} - c \cdot \frac{\delta}{r} \cdot \frac{p'}{p''}\right)$$

Here, again, we have the previously fixed factors P s_z and p' and p'' which, together equal f As above, a section has to be designed having values of A δ and r or I_0 which will satisfy the equation. There remain E and c the latter being a partially empirical factor. At first sight there appear to be several more factors than in the Rankine-Gordon formula, but in that formula the empirical factor a covers all except δ which is the only additional one, and allows for the comparative efficiency of the design.

In this case the value s_z is on the left hand side of the equation instead of P but that makes no difference in the process, which must be exactly as described above. A trial section has to be designed and values of A δ and I_0 inserted in the right hand side of the equation, and the section modified until the resulting value of the right hand side is near enough to the known value of s_z, or L.

The following practical examples illustrate the application of the equations and the processes. When a trial section has been designed it will help much if the values $\frac{\delta}{r}$ and $\frac{L}{r}$ $\left(\text{or } \frac{s_z}{r}\right)$ are first ascertained and a diagram of curves referred to, such as Fig. 48, for the particular material, etc., in fact for most struts a reference to such curves will give results, without further calculation, within a percentage near enough for the purpose. If sets of curves, for various ratios, are drawn to a comparatively large scale they will give results near enough for any purpose whatever, and when certain materials become standardised, the use of the corresponding curves will save considerable time and labour in making calculations.

Example I.

Design a strut having a solid square section, and find the maximum length at which it will carry P without f exceeding 50,000 lb. per sq. in.

Given

$$\text{Type} = \text{No. } 2.$$

$$E = 31{,}000{,}000 \text{ lb.}$$

$$P = 450{,}000 \text{ lb.}$$

$$p' = 25{,}000 \text{ lb. per sq. in.}$$

$$c = \tfrac{1}{10}.$$

We then have

$$A = \frac{P}{p'} = \frac{450{,}000}{25{,}000} = 18 \text{ in.}^2$$

$$D = \sqrt{18} = 4{\cdot}24 \text{ in.}$$

The section is isotropic and therefore I_0 constant.

$$I_0 = \frac{18^2}{12} = 27 \text{ in.}^4$$

by (19),
$$r = \sqrt{\frac{27}{18}} = 1{\cdot}224 \text{ in.}$$

The worst direction for bending is along a diagonal, and

$$\delta = \frac{4{\cdot}24}{\sqrt{2}} = 3 \text{ in.}$$

diagonally,
$$\frac{\delta}{r} = \frac{3}{1{\cdot}224} = \frac{2{\cdot}45}{1}.$$

By (133) max. allowable

$$v = \frac{1{\cdot}224^2}{3}\left(\frac{50{,}000}{25{,}000} - 1\right) = 0{\cdot}5 \text{ in.}$$

By (127) and (145)

$$L = 2\,s_z = 2\sqrt{\frac{31{,}000{,}000 \times 27}{450{,}000}}$$

$$\times \left(1{\cdot}571 - \frac{2{\cdot}45 \times 25{,}000}{10 \times 25{,}000}\right) = 114 \text{ in.}$$

That is,

The area of the section	=	18 in.^{2}
The length of the side of the square	=	4·24 in.
The total length of the strut	=	114 in.

Example II.

Design a strut having a solid square section.

Given

$E = 31{,}000{,}000$ lb.

$f = 50{,}000$ lb. per sq. in.

$P = 450{,}000$ lb.

$L = 200$ in.

$c = \frac{1}{10}.$

Type = No. 3.

The section is isotropic and the worst direction for bending is diagonally. The ratio $\frac{\delta}{r} = \frac{2{\cdot}45}{1}$, as in Ex. 1, and is the same for a square of any size.

By (127) and (145)

$$L = 3\,s_z = 3\sqrt{\frac{E\,I_0}{P}}\left(\frac{\pi}{2} - \frac{1}{10}\cdot\frac{\delta}{r}\cdot\frac{p'}{p''}\right).$$

The unknown factors are I_0 δ r p' and p''. They cannot be determined directly, but as soon as we design a provisional section we obtain trial values for them all. We must first assume a section, and the difficulty is to know what extra material to allow for bending stresses. If there was no bending the area required would be $\frac{P}{f} = \frac{450{,}000}{50{,}000} = 9$ in.^{2} This would give a square 3 in. × 3 in., with r about 0·9 in., giving a ratio $\frac{L}{r} = \frac{200}{0{\cdot}9} = \frac{222}{1}$ for Type 3, or $\frac{s_s}{r} = \frac{74}{1}$. We see from the diagram, Fig. 45, that for these ratios of $\frac{\delta}{r}$ and $\frac{s_s}{r}$, p' would be about 17,000 lb. per sq. in., but that would necessitate A being increased and therefore $\frac{s_s}{r}$ reduced. Make a trial of $\frac{s_s}{r} = \frac{50}{1}$, and the corresponding value of $p' =$ about 23,000 lb. per sq. in.

Then, for first trial,

$$p' = 23{,}000.$$

$$p'' = 27{,}000.$$

$$A = \frac{450{,}000}{23{,}000} = 19{\cdot}6 \text{ in.}^2$$

$$I_0 = \frac{19{\cdot}6^2}{12} = 32 \text{ in.}^4$$

and

$$L = 3\sqrt{\frac{31{,}000{,}000 \times 32}{450{,}000}}\left(1{\cdot}571 - \frac{2{\cdot}45 \times 23{,}000}{10 \times 27{,}000}\right)$$

$$= 192 \text{ in.}$$

This result is too small and A must be increased. As the difference is very small, try increasing A in the ratio of the length required to the length found,
when

$$A = 19{\cdot}6\,\frac{200}{192} = 20{\cdot}4 \text{ in.}^2$$

and take that value for a second trial.

Then

$$A = 20{\cdot}4 \text{ in.}^2$$

$$I_0 = \frac{20{\cdot}4^2}{12} = 34{\cdot}6 \text{ in.}^4$$

$$p' = \frac{450{,}000}{20{\cdot}4} = 22{,}100 \text{ (say)}$$

$$p'' = 50{,}000 - 22{,}100 = 27{,}900.$$

and

$$L = 3\sqrt{\frac{31{,}000{,}000 \times 34{\cdot}6}{450{,}000}}\left(1{\cdot}571 - \frac{2{\cdot}45 \times 22{,}100}{10 \times 27{,}900}\right)$$

$$= 201 \text{ in.}$$

or just a fraction over the amount required.

The results are

Area of section = $20{\cdot}4$ in.^{2}
Side of square = $4{\cdot}52$ in.

Example III.

A hollow Cylindrical Strut of circular section is required. Given

P = 200,000 lb.
Type = No. $3{\cdot}25$
L = 300 in.
E = 31,000,000 lb.
f = 50,000 lb. per sq. in.
Outer diameter D = 6 in.
c = $\frac{1}{10}$

Find the thickness of metal required.

The section is isotropic, and $\delta = \frac{6}{2} = 3$ in.

For a semicircular line, about the diameter, $\frac{\delta}{r}$ is about $\frac{1{\cdot}5}{1}$.

Here

$$\frac{s_z}{r} = \text{about } \frac{300}{3{\cdot}25 \times r} = \text{about } \frac{48}{1},$$

and for these ratios, from Fig. 45, p' will be about 26,000 requiring an area $A = \text{about } \dfrac{200{,}000}{26{,}000} = 7{\cdot}7$ in.^{2}, which would make the metal about 0·45 in. thick.

For a first trial take the internal diameter $d = 5{\cdot}1$ in.

Then

$$I_0 = \frac{\pi}{64}(D^4 - d^4) = \frac{3{\cdot}1416 \times 620}{64} = 30{\cdot}45 \text{ in.}^4$$

Area 6 in. circle − area 5·1 in. circle

$$= 28{\cdot}274 - 20{\cdot}428 = 7{\cdot}846 \text{ in.}^2$$

By (19)

$$r = \sqrt{\frac{30{\cdot}45}{7{\cdot}846}} = 1{\cdot}97 \text{ in.},$$

then

$$\frac{\delta}{r} = \frac{3}{1{\cdot}97} = \frac{1{\cdot}52}{1}$$

$$p' = \frac{200{,}000}{7{\cdot}846} = 25{,}500.$$

$$p'' = 50{,}000 - 25{,}500 = 24{,}500.$$

and, by (127) and (145)

$$L = 3{\cdot}25 \times s_z = 3{\cdot}25\sqrt{\frac{31{,}000{,}000 \times 30{\cdot}45}{200{,}000}}$$

$$\times \left(1{\cdot}571 - \frac{1{\cdot}52 \times 25{,}500}{10 \times 24{,}500}\right) = 315 \text{ in.}$$

This result is too high. The ratio of variation of A in relation to L is not simple. For a second approximation try $A = \text{about } 7{\cdot}3$ in.^{2}

If we make

$$d = 5{\cdot}17 \text{ in.}$$

then

$$A = 28{\cdot}274 - 20{\cdot}99 = 7{\cdot}28 \text{ in.}^2$$

$$I_0 = \frac{3{\cdot}1416 \times 58^2}{64} = 28{\cdot}6 \text{ in.}^4$$

$$r = \sqrt{\frac{28{\cdot}6}{7{\cdot}28}} = 1{\cdot}98 \text{ in.}$$

$$\frac{\delta}{r} = \frac{3}{1{\cdot}98} = \frac{1{\cdot}515}{1}.$$

$$p' = \frac{200{,}000}{7{\cdot}28} = 27{,}500.$$

$$p'' = 50{,}000 - 27{,}500 = 22{,}500.$$

and

$$L = 3{\cdot}25 \sqrt{\frac{31{,}000{,}000 \times 28{\cdot}6}{200{,}000}}$$

$$\times \left(1{\cdot}571 - \frac{1{\cdot}515 \times 27{,}500}{10 \times 22{,}500}\right) = 299{\cdot}9 \text{ in.}$$

which result is near enough for any purpose.

The required thickness of metal is

$$\frac{6 - 5{\cdot}17}{2} = 0{\cdot}415 \text{ in.}$$

The ratio of $\frac{\text{radius of curvature}}{\text{thickness of metal}} = \frac{3}{0{\cdot}415} = \frac{7{\cdot}2}{1}$, therefore, from (153), there is no possibility of local flexure.

Example IV.

A column is to be constructed of two channel sections placed with their flanges turned towards the axis and at such a distance apart that I_0 is the same in all directions.

Given, section and dimensions of channel as in Fig. 101.

$$A = 2 \times 9{\cdot}76 = 19{\cdot}34 \text{ in.}^2$$
$$I_{OY} = 2 \times 191 = 382 \text{ in.}^4$$
$$I_{OX_1} = 8{\cdot}92 \text{ in.}^4\text{, each channel.}$$
$$\text{Type} = \text{No. 2.}$$
$$L = 600 \text{ in.}$$
$$f = 50{,}000 \text{ lb. per sq. in.}$$
$$c = \frac{1}{10}$$

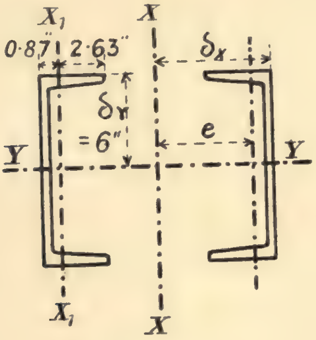


FIG. 101.

Find, (1) the distance apart at which the channels must be placed. Call the distance from X X to $X_1 X_1 = e$.

$$I_{OX} \text{ must equal } I_{OY} = 382 \text{ in.}^4$$

$$I_{OX} = 2 \times I_{OX_1} + 2 \text{ (area of channel) } e^2$$

$\therefore$

$$e^2 = \frac{\frac{I_{OX}}{2} - I_{OX_1}}{\text{area of channel}} = \frac{191 - 8{\cdot}92}{9{\cdot}67} = 18{\cdot}85,$$

and

$$e = \sqrt{18{\cdot}85} = 4{\cdot}34 \text{ in.}$$

Then, from the figure,

$$\delta_X = 4{\cdot}34 + 0{\cdot}87 = 5{\cdot}21 \text{ in.}$$

which is the distance required.

Find (2), the safe working load, with f.o.s. = 4, taking it that the channels are properly braced together and that there is no local flexure.

We have $\delta_X = 5{\cdot}21$ and $\delta_Y = 6$, and as the section has been made isotropic the worst direction for bending is along a diagonal, in which direction

$$\delta = \sqrt{6^2 + 5{\cdot}21^2} = 7{\cdot}96 \text{ in.}$$

By (19)

$$r = \sqrt{\frac{382}{19{\cdot}34}} = 4{\cdot}45 \text{ in.}$$

$$\frac{\delta}{r} = \frac{7{\cdot}96}{4{\cdot}45} = \frac{1{\cdot}79}{1}.$$

$$\frac{s_z}{r} = \frac{600}{2 \times 4{\cdot}45} = \frac{67{\cdot}5}{1}.$$

For these ratios, by Fig. 45, p' will be about 15,000 lb. per sq. in. and

$$p'' = 50{,}000 - 15{,}000 = 35{,}000 \text{ lb. per sq. in.}$$

$$\text{P} = 15{,}000 \times 19{\cdot}34 = 290{,}100 \text{ lb.}$$

then, by (127) and (145)

$$\text{L} = 2\,s_z = 2\sqrt{\frac{31{,}000{,}000 \times 382}{290{,}100}} \times \left(1{\cdot}571 - \frac{1{\cdot}79 \times 15{,}000}{10 \times 35{,}000}\right) = 603 \text{ in.}$$

which result is probably near enough. If a closer result is desired call P = 290,400 lb. and it will be found to give almost exactly L = 600.

Then

$$p' = 15{,}016 \quad\text{and}\quad p'' = 34{,}984.$$

The safe working load is

$$\frac{290{,}100}{4} = 72{,}600 \text{ lb.}$$

Example V.

Find approximately the worst direction of bending or deflection for the section of chord A. 9 of the Quebec Bridge, as Fig. 94, and details in Chapter XII.

Given

$c = \frac{1}{10}$
$I_{OY} = 311{,}100$ in.^{4} ; and $\sqrt{I_{OY}} = 558$.
$\delta_Y = 33{\cdot}75$ in.
$r_Y = 19{\cdot}97$ in.
$I_{OX} = 202{,}200$ in.^{4} ; and $\sqrt{I_{OX}} = 450$.
$\delta_X = 27$ in.
$r_X = 16$ in.
$A = 780$ in.^{2} ; and $\sqrt{A} = 28$.
$L = 684$ in.

Take the Type as No. 2·2 ; then $s_z = \dfrac{684}{2\cdot 2} = 311$ in.

We have $\frac{s_z}{r}$ about $\frac{17}{1}$ and a reference to Fig. 45, shows that $\frac{p'}{p''}$ will be about $\frac{4}{1}$.

Turn now to Fig. 48, which represents this case and is drawn to scale. The rectangle *b c e d* is the figure enclosing the section. Then follow the method described in relation to that figure. The quadrant of the ellipse may be drawn by any convenient method. The distance O *c* is the maximum value of δ and is

$$\delta = \sqrt{\delta_Y^2 + \delta_X^2} = \sqrt{33{\cdot}75^2 + 27^2} = 43 \text{ in.}$$

then

$$\left(\frac{2c\sqrt{A}}{\pi} \cdot \frac{p'}{p''}\right)\delta = \left(\frac{2 \times 28 \times 4}{10 \times 3{\cdot}1416}\right) 43 = 306.$$

This is the diameter of the circle O *r s* with centre at *p* drawn to the same scale as the ellipse of inertia.

Then, by trial, the vector O m is found so that $r\,m$ is a minimum. The direction O m is that required.

By scaling from the figure, which, for practical purposes, is drawn to a larger scale than in Fig. 48, we find that (in the direction O m) $\delta = 40{\cdot}3$; and $I_0 = 233{,}300$. whence

$$r = 17{\cdot}3\ ;\ \text{and}\ \frac{\delta}{r} = \frac{2{\cdot}33}{1}.$$

Example VI.

What is the safe load, allowing for a factor of safety of 3, for a strut having a section as in Ex. V., which is the section of chord A·9 of the Quebec Bridge. Assume that proper provision is made for avoiding local flexure, and allow for main stresses only ; do not allow for the weight of the strut itself.

Take the ultimate strength of the material as $f = 60{,}000$ lb. per sq. in., which was the compressive strength specified for the steel in the bridge.

Given

$E = 31{,}000{,}000$ lb.
Type = No. 2·2.
$L = 684$ in. ; whence $s_z = 311$ in.
$c = \frac{1}{10}$
$A = 780$ in.^{2}

From Ex. V. we have, for the worst direction of bending,

$I_0 = 233{,}300$ in.^{4}
$\delta = 40{\cdot}3$ in.
$r = 17{\cdot}3$ in.
$\frac{\delta}{r} = \frac{2{\cdot}33}{1}$.

We have no curves drawn for $f = 60{,}000$. By Fig. 45, as a guide, for $f = 50{,}000$; p' would be about 40,000. This will not be increased in the same ratio as f For first trial take

$$p' = 47{,}000; \text{ then } p'' = 13{,}000;$$

and

$$P = 36{,}660{,}000.$$

then, by (127) and (145)

$$L = 2\cdot2\sqrt{\frac{31{,}000{,}000 \times 233{,}300}{36{,}660{,}000}} \times \left(1\cdot571 - \frac{2\cdot33 \times 47{,}000}{10 \times 13{,}000}\right) = 712 \text{ in.}$$

a result which is too great. At these low ratios of $\frac{s_x}{r}$ a small difference in p' makes a considerable difference in s_x. For second trial take

$$p' = 47{,}300; \quad p'' = 12{,}700;$$

and

$$P = 36{,}894{,}000$$

then

$$L = 2\cdot2\sqrt{\frac{31{,}000{,}000 \times 233{,}300}{36{,}894{,}000}} \times \left(1\cdot571 - \frac{2\cdot33 \times 47{,}300}{10 \times 12{,}700}\right) = 684\cdot75 \text{ in.}$$

or almost exactly the result required.

The safe load is then

$$\frac{36{,}894{,}000}{3} = 12{,}298{,}000 \text{ lb.}$$

Note that when the bridge was designed, this strut was intended to take a working load of $p' = 24{,}000$ lb. per sq. in., making $P = 24{,}000 \times 780 = 18{,}720{,}000$ lb.

This would give a factor of safety, on the ultimate load above, of

$$\frac{36{,}894{,}000}{18{,}720{,}000} = 1\cdot97.$$

Example VII.

Find the ultimate load P on the strut of Ex. VI., under similar conditions, but if constrained to bend about the horizontal axis X X in Fig. 94.

We have all data as before except

$$I_0 = 202{,}200 \text{ in.}^4$$
$$r = 16 \text{ in.}$$
$$\frac{\delta}{r} = \frac{27}{16} = \frac{1 \cdot 68}{1}.$$
$$\frac{s_z}{r} = \frac{311}{16} = \frac{19 \cdot 4}{1}.$$

The strut should carry a somewhat higher load than if it deflected in the worst direction. For a first trial take

$$p' = 50{,}000 \, ; \; p'' = 10{,}000 \, ; \; \text{P} = 39{,}000{,}000 \text{ lb.}$$

then

$$\text{L} = 2 \cdot 2 \sqrt{\frac{31{,}000{,}000 \times 202{,}200}{39{,}000{,}000}} \left(1 \cdot 571 - \frac{1 \cdot 68 \times 50{,}000}{10 \times 10{,}000}\right)$$
$$= 644 \text{ in.}$$

which is too small. Try next

$$p' = 49{,}500 \, ; \; p'' = 10{,}500 \, ; \; \text{P} = 38{,}610{,}000 \text{ lb.}$$

then

$$\text{L} = 2 \cdot 2 \sqrt{\frac{31{,}000{,}000 \times 202{,}200}{38{,}610{,}000}} \left(1 \cdot 571 - \frac{1 \cdot 68 \times 49{,}500}{10 \times 10{,}500}\right)$$
$$= 690 \text{ in.}$$

which is quite near enough to the required value of 684 in., as P varies at a less ratio than s_z in this case.

Comparing this result with that of Ex. VI., we find that the ultimate load for bending about X X is greater than that in the worst direction, as found, in the ratio of

$$\frac{38{,}610{,}000}{36{,}894{,}000} = \frac{1 \cdot 048}{1}, \text{ or } 4 \cdot 8 \text{ per cent.}$$

Example VIII.

Find the ultimate loads on the strut of Ex. IV. (1) if constrained to bend about the axis Y Y and (2) if about X X and compare the results with that for the worst direction of flexure, already found.

(1) We have all factors as in Ex. IV. except

$$\delta = 6 \text{ in.}$$
$$\frac{\delta}{r} = \frac{6}{4{\cdot}45} = \frac{1{\cdot}35}{1}.$$

For first trial take $p' = 15{,}500$; $p'' = 34{,}500$.

Then

$$P = 15{,}500 \times 19{\cdot}34 = 299{,}770 \text{ lb.}$$

and

$$L = 2\sqrt{\frac{31{,}000{,}000 \times 382}{299{,}770}}\left(1{\cdot}571 - \frac{1{\cdot}35 \times 15{,}500}{10 \times 34{,}500}\right)$$
$$= 600 + \text{in.}$$

or practically the figure required.

Then, comparing with P for worst direction,

$$\frac{299{,}770}{290{,}400} = \frac{1{\cdot}032}{1},$$

or 3·2 per cent. stronger about Y Y than in worst direction.

(2) About X X we have

$$\delta = 5{\cdot}21 \text{ in.}$$
$$\frac{\delta}{r} = \frac{5{\cdot}21}{4{\cdot}45} = \frac{1{\cdot}17}{1}.$$

For these ratios try $p' = 15{,}800$; $p'' = 34{,}200$.

Then

$$P = 15{,}800 \times 19{\cdot}34 = 305{,}570 \text{ lb.}$$

and

$$L = 2\sqrt{\frac{31{,}000{,}000 \times 382}{305{,}570}}\left(1{\cdot}571 - \frac{1{\cdot}17 \times 15{,}800}{10 \times 34{,}200}\right)$$

$$= 597 \text{ in.}$$

which is slightly too small.

Try

$$p' = 15{,}700; \quad p'' = 34{,}300.$$

Then

$$P = 15{,}700 \times 19{\cdot}34 = 303{,}640,$$

and

$$L = 2\sqrt{\frac{31{,}000{,}000 \times 382}{303{,}640}}\left(1{\cdot}571 - \frac{1{\cdot}17 \times 15{,}700}{10 \times 34{,}300}\right)$$

$$= 600 - \text{in.}$$

This compared with the worst direction is

$$\frac{303{,}640}{290{,}400} = \frac{1{\cdot}045}{1} \text{ or } 4{\cdot}5 \text{ per cent. stronger.}$$

Example IX.

In Ex. VIII. we had, for the double channel section, bending about the axis X X ; P = 303,640 lb., etc.

If the two channels are connected together by two sets of diagonal bracings, one set between each pair of opposite flanges, and of the type shown in Fig. 81 (*a*), upper half, the angle between the diagonal bar and the axis of the strut being $\psi = 60°$; find the ultimate stress, for which the bars and the rivets attaching them, should be proportioned.

We have, as before, $p' = 15{,}700$ lb. per sq. in.

By (166)

The stress on each bracing bar $= \dfrac{R_s}{2 \times \sin\psi}$, there being two sets.

By (165)

$$R_s = 2 \cdot 5 \frac{P v}{S}.$$

By (134)

$$v = \frac{I_0}{\delta}\left(\frac{f}{P} - \frac{1}{A}\right) = \frac{382}{5 \cdot 21}\left(\frac{50{,}000}{303{,}640} - \frac{1}{19 \cdot 34}\right) = 8 \cdot 3 \text{ in.}$$

By (110)

$$S = \sqrt{\frac{E I_0}{P}} \cdot \frac{\pi}{2} = \sqrt{\frac{31{,}000{,}000 \times 382}{303{,}640}} \times 1 \cdot 571$$
$$= 310 \text{ in.}$$

Then stress on each bar

$$= \frac{2 \cdot 5 \times 303{,}640 \times 8 \cdot 3}{2 \times 0 \cdot 866 \times 310} = 11{,}735 \text{ lb.}$$

Example X.

In chord A·9 of the Quebec Bridge the bracing was of the type of Fig. 82, the diagonals being at an angle of 45°. Take P at 36,894,000 lb. as in Ex. VI. Assume that the diagonal and transverse bars are to be of the same gross sectional areas. Determine a suitable section for these bars.

This is probably one of the most difficult and tedious problems in strut design. The load P is less than the ultimate strength of the strut in the direction of the bracing. The deflection due to that value of P cannot be exactly determined, and the stresses due to main flexure depend upon the amount of the deflection. The bars are stressed by the direct thrust of P and the amounts of the stresses depend upon the gross sections of the bars. The process of finding each component of the resulting stresses must be a tentative one.

To find the stresses due to main flexure we must determine some value for v By trial we find that the ultimate load in this direction is about 39,670,000 lb.

The deflection due to this load is, by (134)

$$v = \frac{I_0}{\delta}\left(\frac{f}{P} - \frac{1}{A}\right) = \frac{311{,}100}{33{\cdot}75}\left(\frac{60{,}000}{39{,}670{,}000} - \frac{1}{780}\right)$$
$$= 2{\cdot}125 \text{ in.}$$

In this case the value of P in the direction of the bracing is only about 7 per cent. greater than in the weakest direction, and we may reduce the deflection to a somewhat greater extent. Let $v = 2$ in.

By (110),

$$S = \sqrt{\frac{31{,}000{,}000 \times 311{,}100}{39{,}670{,}000}} \times 1{\cdot}571 = 775 \text{ in.}$$

Then by (165) and (166), there being 4 sets of diagonal bars.

$$\text{Stress on each bar} = \frac{2{\cdot}5 \times 36{,}894{,}000 \times 2}{4 \times 0{\cdot}707 \times 775}$$
$$= 84{,}170 \text{ lb.}$$

We cannot find any stresses due to the direct thrust P until the sectional areas of the bars are known. To assist in estimating the areas for trial see the notes at end of Chapter X. As the bars are at 45°, the direct stress intensity will be about one third of that in the main members.

By Ex. VI. the value of p' for these members = 47,300.

Then $\frac{47{,}300}{3}$ = about 16,000 lbs. per sq. in., and if we take the ultimate value of p' to be about the same for the bracing bars there is a margin of

$$47{,}300 - 16{,}000 = \text{say } 31{,}000 \text{ lb. per sq. in.}$$

to take the stress due to bending.

The sectional area required is then about

$$\frac{84{,}170}{31{,}000} = \text{about } 2{\cdot}72 \text{ sq. in.}$$

Use that value, multiplied by 2, as there are two sets of bracings, for a trial value for β and γ in equations (172), (173) and (174). Then, due to the direct action of P

$$\text{Stress } \beta = \frac{0{\cdot}5 \times 36{,}894{,}000}{\frac{390}{0{\cdot}707^2}\left(\frac{1}{5{\cdot}44} + \frac{2 \times 0{\cdot}707^3}{5{\cdot}44}\right) + 0{\cdot}707}$$

$$= 75{,}140 \text{ lb.}$$

$$\text{Stress } \gamma = 2 \times 75{,}140 \times 0{\cdot}707 = 106{,}200 \text{ lb.}$$

$$\text{Stress } \alpha = 18{,}447{,}000 - 75{,}140 \times 0{\cdot}707 = 18{,}393{,}900 \text{ lb.}$$

It will be seen that the main members are only very slightly relieved of stress.

The direct stress on each diagonal bar is

$$\frac{75{,}140}{2} = 37{,}570 \text{ lb.}$$

and adding that due to bending, we have the total stress on the bar

$$37{,}570 + 84{,}170 = 121{,}740 \text{ lb.}$$

The stress intensity on the gross area is then

$$\frac{121{,}740}{2{\cdot}72} = 44{,}800 \text{ lb. per sq. in.}$$

This is slightly less than the value of 47,300 aimed at and, if desired, the 2·72 sq. in. might be slightly reduced, but the difference would be small.

The stresses due to bending may be either tensile or compressive. The total value above is the maximum compressive stress. The oppositely inclined bars, at the same time would carry tensile stresses amounting, in each bar, to

$$84{,}170 - 37{,}570 = 46{,}600 \text{ lb.}$$

The diagonal bars, then, would have to be so proportioned that, acting as struts, they would carry the ultimate load of 121,740 lb. each. The rivets at their ends would also have to be sufficient in number to resist that stress. If the thickness of metal was 0·375 in. and the rivets 0·875 in. in diameter, as they were in the bridge, then, taking the ultimate bearing pressure as 60,000 lb. per sq. in., the same as the ultimate strength of the steel, the number of rivets required would be

$$\frac{121{,}740}{0.375 \times 0{\cdot}875 \times 60{,}000} = \text{at least 6 rivets for each bar,}$$

and that number, in single shear, would be sufficient to carry the shearing stresses also.

The stresses in the transverse bars are tensile. They are not stressed by the bending action and the total stress in each bar is then, as there are two bars,

$$\frac{106{,}200}{2} = 53{,}100 \text{ lb.}$$

and being tensile the intensity must be calculated on the net area. That area depends upon the proportion cut away by rivet holes, etc. If it was not possible to get the net area required without increasing the gross area, the whole of the calculations would have to be re-made, allowing for the altered areas. As the total stress is only about half that on a diagonal, the net area is certain to be sufficient.

Example XI.

A parallel strut is formed of 4 booms connected by diagonal bracings. The section of the strut is square and the booms placed at the corners.

For each separate boom we have given

Free length between bracing attachments $l = 66$ in.
Sectional area $\frac{A}{4} = 4$ in.^{2}
Moment of Inertia $I_0 = 5$ in.^{4}
In worst direction $\begin{cases} \text{radius of gyration} & r = 0{\cdot}8 \text{ in.} \\ \text{Extreme fibre from axis} & \delta = 2 \text{ in.} \end{cases}$

For the whole section Type = No. 2.
$L = 700$ in.
$A = 16$ in.^{2}
$I_0 = 1600$ in.^{4}
$r = 10$ in.
Diagonally, $\delta = 14$ in.

And generally

$E = 31{,}000{,}000$ lb. = higher limit for compression.
$f = 50{,}000$ lb. per sq. in. = ultimate compressive strength.
$c = \frac{1}{10}$

Find the ultimate load P for the whole strut, on the understanding that all bracings are correct and that only stresses due to the main action of P have to be considered.

In this case the unsupported length of the booms between the bracing attachments is comparatively large, and the strength of each boom must first be ascertained. By (150) it is not necessary to allow for a deviation of the thrust from the axis, and as the bracings are to be properly designed they will intersect on the axis and in each boom the load may be taken as axial. We then have

$$f_1 = \frac{E\,\pi^2}{\left(\frac{l}{r}\right)^2} = \frac{31{,}000{,}000 \times \pi^2}{\left(\frac{66}{0 \cdot 8}\right)^2} = 45{,}000 \text{ lb. per sq. in.}$$

This value of f_1 has to be substituted for f in calculating the strength of the strut as a whole, because when the stress intensity in the boom due to the direct action of P and to main flexure amounts to f_1 the local bending between the bracing attachments will increase f_1 to f at one edge of the section of the boom.

For the whole strut we have, by (127) and (145)

$$L = 2\,s_z = 2\sqrt{\frac{E\,I_0}{P}}\left(\frac{\pi}{2} - c\,.\,\frac{\delta}{r}\cdot\frac{p'}{p''}\right)$$

As before, we have to proceed tentatively and assume a value of P for trial.

$$\text{As } \frac{\delta}{r} = \frac{14}{10} = \frac{1 \cdot 4}{1}, \text{ and } \frac{L}{r} = \frac{700}{10} = \frac{70}{1},$$

we find from the diagram (Fig. 45) that p' would be about 37,000 lb. for Type 2, and $f = 50{,}000$. We have, however, reduced f to f_1 (about 10 per cent.), therefore try $p' = 33{,}500$ when P will be 536,000 and $p'' = 11{,}500$. Then

$$L = 2\sqrt{\frac{31{,}000{,}000 \times 1600}{536{,}000}}\left(\frac{\pi}{2} - \frac{1 \cdot 4 \times 33{,}500}{10 \times 11{,}500}\right)$$
$$= 707 \text{ in.}$$

If a nearer result is desired, p' must be slightly reduced, but the trial value of 536,000 lb. is quite near enough for any ordinary purposes.

In this case, if the distance l between the bracing attachments was reduced from 66 in. to something less than 60 in., the value of f would not be appreciably reduced by local flexure. This example is, of course, given as an extreme case. As the side of the square forming the main section of the strut is only about 20 in. it would hardly be practicable to make l as great as 60 in. and in such a strut it would usually not be greater than 30 or 40 in.

It is not possible to give practical examples containing all conditions that may arise, but the foregoing serve to illustrate the general application of the main equations, etc. Such questions as bending under own weight, and all local actions, have always to be considered as met with in each individual case.

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